ALGORITHMS FOR EXPERIMENTING WITH ZARISKI DENSE SUBGROUPS

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ABSTRACT. We give a method to describe all congruence images of a finitely generated Zariski dense group $H \leq \mathrm{SL}(n, \mathbb{Z})$. The method is applied to obtain efficient algorithms for solving this problem in odd prime degree n; if $n = 2$ then we compute all congruence images only modulo primes. We propose a separate method that works for all n as long as H contains a known transvection. The algorithms have been implemented in GAP, enabling computer experiments with important classes of linear groups that have recently emerged.

1. INTRODUCTION

This paper further develops methods and algorithms for computing with linear groups over infinite domains. It is a sequel to [7].

Let H be a finitely generated subgroup of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, that is Zariski dense in $\mathrm{SL}(n, \mathbb{R})$. By the strong approximation theorem, H is congruent to $\mathrm{SL}(n, p)$ modulo p for all but a finite number of primes p [23, p. 391]. If $n > 2$ and H is arithmetic, i.e., H has finite index in $\mathrm{SL}(n, \mathbb{Z})$, then the congruence subgroup property guarantees that H contains a *principal congruence subgroup of level* m for some m, i.e., the kernel $\Gamma_{n,m}$ of the reduction modulo m homomorphism $\varphi_m : \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}_m)$. In that event H contains a unique maximal principal congruence subgroup $\Gamma_{n,M}$, and we call $M = M(H)$ the *level of H*. The dense group H is contained in a uniquely defined minimal arithmetic overgroup $\mathrm{cl}(H)$, namely the intersection of all arithmetic subgroups of $\mathrm{SL}(n, \mathbb{Z})$ containing H (its 'arithmetic closure') [7, Section 3.3]. The level of H is defined to be the level of $\mathrm{cl}(H)$, and is again denoted $M(H)$. Sarnak [29] calls dense non-arithmetic $H \leq \mathrm{SL}(n, \mathbb{Z})$ a *thin matrix group*.

In [7] we developed practical algorithms to compute the level M of a dense group $H \leq \mathrm{SL}(n, \mathbb{Z})$ for $n > 2$. This was motivated by the fact that M is the key component of our algorithms to compute with arithmetic subgroups of $\mathrm{SL}(n, \mathbb{Z})$ [6]. Once we have M, we can find $\mathrm{cl}(H)$ and obtain further information about dense H via computation with $\mathrm{cl}(H)$.

Algorithms to compute M were implemented and used to carry out extensive computer experiments, as detailed in [7, Section 4]. Our method requires the set $\Pi = \Pi(H)$ of primes p such that $\varphi_p(H) \neq \mathrm{SL}(n, p)$: essentially, a computational realization of the strong approximation theorem.

The aim of the present work is twofold. First, in Section 2 we establish a general method to compute $\Pi(H)$ for dense $H \leq \mathrm{SL}(n, \mathbb{Z})$, based on the classification of maximal subgroups of $\mathrm{SL}(n, p)$ as in [2] (see also [23, p. 397]). This is then applied in Section 3 to obtain efficient algorithms to compute $\Pi(H)$ for prime degree n (in which case the types of maximal subgroups of $\mathrm{SL}(n, p)$ are quite restricted). Moreover, for odd prime n, we build on this knowledge to describe the congruence images of H modulo all positive integers. Arbitrary degrees n are treated in [9] (albeit with algorithms that are less efficient for prime n than those herein).

We also give an algorithm to compute $\Pi(H)$ for subgroups H of $\mathrm{SL}(2n, \mathbb{Z})$ that contain a known transvection (a unipotent element t such that $t - 1_n$ has matrix rank 1). This completes the task begun in [7, Section 3.2].

Another goal is to perform computer experiments successfully with low-dimensional dense representations of finitely presented groups that have recently been the focus of much attention. We compute Π and M for each group, thus enabling us to describe all of its congruence quotients. Experimental results are presented in Section 4.

We adhere to the following conventions and notation. Congruence images are sometimes indicated by overlining. Pre-images in $H = \langle g_1, \ldots, g_r \rangle \leq \mathrm{SL}(n, \mathbb{Z})$ of elements of $\bar{H}$ written as words in the $\bar{g}_i$ are found by 'lifting': $\bar{g}_{k_1}^{m_1} \cdots \bar{g}_{k_s}^{m_s}$ has pre-image $g_{k_1}^{m_1} \cdots g_{k_s}^{m_s}$. The set of prime divisors of $a \in \mathbb{N}$ is denoted $\pi(a)$. Throughout, $\mathbb{F}$ is a field.

2. STRONG APPROXIMATION AND RECOGNITION OF CONGRUENCE IMAGES

The core idea of our approach to computing $\Pi(H)$ is to find all primes p such that $\varphi_p(H)$ lies in a maximal subgroup of $\mathrm{SL}(n, p)$. Here we provide general methods for this purpose.

2.1. **Large congruence images.** Let H be infinite. Given a positive integer k, we find all primes p such that $\varphi_p(H)$ has elements of order greater than k (cf. [32, Chapter 4] and [10, Section 3.5]).

Since a periodic linear group is locally finite, the finitely generated group H has an element h of infinite order. We can find h quickly by random selection (see [10, Section 4.2, p. 107], and the discussion in Subsection 3.3 on randomly selecting elements with specified properties). For $1 \leq i \leq k$, let m_i be the greatest common divisor of the non-zero entries of $h^i - 1_n$, and let $l = \mathrm{lcm}(m_1, \ldots, m_k)$. If $p \notin \pi(l)$ then $|\varphi_p(H)| > k$. For each $p \in \pi(l)$ we check whether $|\varphi_p(H)| < k$. The preceding steps define a procedure `PrimesForOrder` that accepts k and infinite $H \leq \mathrm{SL}(n, \mathbb{Z})$, and returns the (finite) set of primes p such that $|\varphi_p(H)| < k$.

We will also need the following.

Lemma 2.1. *Suppose that $\varphi_p(H) = \mathrm{SL}(n, p)$ for some prime p.*

   (i) *If $n \geq 3$ then H is infinite.*
   (ii) *If $n = 2$ and $p \geq 3$ then H is infinite.*

Proof. Theorem A of [12] states the largest order of a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$. In both cases (i) and (ii), this maximal order is less than $|\mathrm{SL}(n, p)|$. $\square$

2.2. **Irreducibility.** This subsection recaps an argument from [7, Section 3.2].

We test whether $H \leq \mathrm{SL}(n, \mathbb{Z})$ is absolutely irreducible by computing a $\mathbb{Q}$-basis $\mathcal{A} = \{A_1, \ldots, A_m\}$ of the enveloping algebra $\langle H \rangle_{\mathbb{Q}}$, where the A_i are words over a generating set of H. If $m = n^2$ then H is absolutely irreducible, and $\varphi_p(H)$ is absolutely irreducible for any prime p not dividing $\Delta := \det[\mathrm{tr}(A_i A_j)]$; here $\mathrm{tr}(x)$ is the trace of a matrix x. Hence we have the following.

Lemma 2.2. *If H is absolutely irreducible then $\varphi_p(H)$ is absolutely irreducible for almost all primes p.*

If $p \,|\, \Delta$ then $\varphi_p(H)$ might be absolutely irreducible. Testing for this is the last step in `PrimesForAbsIrreducible(H)`, which returns the set of all primes p such that $\varphi_p(H)$ is not absolutely irreducible.

Note that if $\bar{H}$ is absolutely irreducible (e.g., $\bar{H} = \mathrm{SL}(n, p)$), and $\bar{\mathcal{A}}$ is a basis of $\langle \bar{H} \rangle_{\mathbb{Z}_p}$, then $\mathcal{A}$ is a basis of $\langle H \rangle_{\mathbb{Q}}$.

2.3. **Primitivity.** Next, we give conditions for the congruence image of an (irreducible) primitive subgroup of $\mathrm{SL}(n, \mathbb{Z})$ to be imprimitive. The main concern is prime n; in such degrees an irreducible linear group is either primitive or monomial.

Lemma 2.3. *If $H \leq \mathrm{SL}(n, \mathbb{Z})$ is not solvable-by-finite then $\varphi_p(H)$ is not monomial for almost all primes p.*

Proof. Since H has a free non-abelian subgroup by the Tits alternative, given $k \geq 1$ there exist $g, h \in H$ such that $c := [g^k, h^k] \neq 1_n$. Then $[\bar{g}^k, \bar{h}^k] = \bar{c} \neq 1_n$ for almost all primes p. The lemma follows by taking k to be the exponent of $\mathrm{Sym}(n)$. $\square$

Lemma 2.4. *For prime n, an infinite solvable-by-finite primitive (irreducible) subgroup H of $\mathrm{SL}(n, \mathbb{Z})$ is solvable.*

Proof. Let $K \trianglelefteq H$ be solvable of finite index. Since n is prime, K is scalar or irreducible. If K were scalar then H would be finite. Thus K is irreducible. If K were monomial over $\mathbb{Q}$ then it would be finite once more; so K is primitive.

Let A be a maximal abelian normal subgroup of K. Then A is irreducible, and $K = \langle A, g \rangle$ for some g because the field $\langle A \rangle_{\mathbb{Q}}$ is a cyclic extension of $\mathbb{Q}1_n$ of degree n. If H normalizes $\langle A \rangle_{\mathbb{Q}}$ then H is solvable. Suppose on the contrary that $hah^{-1} = bg^k$ for some $h \in H$, $a, b \in A$, and k coprime to n. Conjugation by bg^k induces a $\mathbb{Q}$-automorphism of $\langle A \rangle_{\mathbb{Q}}$ that fixes $(bg^k)^n$. Hence $(bg^k)^n$ is scalar, implying that a has finite order. But there is an infinite order element in A. We conclude that H must normalize $\langle A \rangle_{\mathbb{Q}}$. $\square$

Corollary 2.5. *Let n be prime. If $H \leq \mathrm{SL}(n, \mathbb{Z})$ is infinite, non-solvable, and primitive, then $\varphi_p(H)$ is primitive for almost all primes p.*

Given an input group H that is not solvable-by-finite, `PrimesForMonomial` returns the set of primes p such that $\varphi_p(H)$ is monomial. The proof of Lemma 2.3 furnishes a method to compute this finite set. First we find $g, h \in H$ such that $[g^k, h^k] \neq 1_n$, where k is the exponent of $\mathrm{Sym}(n)$. (In our experiments g, h are found by random selection; cf. [1],

and see Subsection 3.3.) Let d be the gcd of the non-zero entries of $[g^k, h^k] - 1_n$. Then $\varphi_p(H)$ is non-monomial if $p \notin \pi(d)$. Finally, we test whether $\varphi_p(H)$ is monomial for each $p \in \pi(d)$, using, e.g., [26].

Although we can detect whether H has a free non-abelian subgroup [11], we do not have an algorithm to locate one. Indeed, as far as we know, the problem of deciding freeness of a finitely generated linear group is not known to be decidable.

2.4. **Solvability.** Zassenhaus's theorem [31, p. 136] implies existence of a bound $\delta = \delta(n)$ on the derived length of solvable subgroups of $\mathrm{SL}(n, \mathbb{F})$ that depends only on n, not on $\mathbb{F}$. See, e.g., [31, p. 136] for an estimate of δ due to Dixon.

Let $H \leq \mathrm{SL}(n, \mathbb{Z})$ be non-solvable. We sketch a procedure $\texttt{PrimesForSolvable}(H, \delta)$ that returns the set of primes p such that $\varphi_p(H)$ is solvable and $\varphi_p(H) \neq \mathrm{SL}(n, p)$. Take a non-trivial iterated commutator in H. As usual, we do this by random selection in H, or by lifting to H from a (non-solvable) congruence image: pick $[\bar{h}_1, \ldots, \bar{h}_{\delta+1}] \neq 1_n$ in $\bar{H}$; then $g = [h_1, \ldots, h_{\delta+1}] \in H$ is as required. Let d be the gcd of the non-zero entries of $g - 1_n$. Then $\varphi_p(H)$ is non-solvable if $p \notin \pi(d)$. Solvability of $\varphi_p(H)$ for $p \in \pi(d)$ can be tested using [26]. We have proved the following.

Lemma 2.6. *If H is non-solvable then $\varphi_p(H)$ is non-solvable for almost all primes p.*

We get better bounds on derived length for irreducible groups in prime degree.

Lemma 2.7. *Let n be prime. An irreducible solvable subgroup G of $\mathrm{GL}(n, \mathbb{F})$ has derived length $d \leq 6$.*

Proof. A monomial group G is an extension of its subgroup of diagonal matrices by a solvable transitive permutation group of prime degree. Such permutation groups are metacyclic, so $d \leq 3$. Suppose that G is primitive. By [32, Theorem 3.3, p. 42], there exists $E \trianglelefteq G$ of derived length at most 2, such that G/E is isomorphic to a subgroup of $\mathrm{SL}(2, n)$. Since $\delta(\mathrm{SL}(2, n)) \leq 4$ (see, e.g., [31, §21.3]), we get $d \leq 6$ as required. $\qquad\square$

Remark 2.8. If $n = 2, 3$ and $G \leq \mathrm{SL}(n, \mathbb{F})$ then $d \leq 4$, $d \leq 5$, respectively.

2.5. **Isometry.** We say that $G \leq \mathrm{GL}(n, \mathbb{F})$ is an *isometry group* if it preserves a non-degenerate bilinear (symmetric or alternating) form. On the other hand, since $\mathrm{SL}(2, \mathbb{F}) = \mathrm{Sp}(2, \mathbb{F})$, we say that G is not an isometry group if G does not preserve a non-degenerate bilinear form for $n > 2$.

Lemma 2.9. *Let $G \leq \mathrm{GL}(n, \mathbb{F})$ be absolutely irreducible. Then G is an isometry group if and only if $\mathrm{tr}(g) = \mathrm{tr}(g^{-1})$ for all $g \in G$.*

Proof. Suppose that $\mathrm{tr}(g) = \mathrm{tr}(g^{-1})$ for all $g \in G$. As their characters are equal, the identity and contragredient representations of G are therefore equivalent; i.e., $g = \Phi(g^\top)^{-1}\Phi^{-1}$ for some $\Phi \in \mathrm{GL}(n, \mathbb{F})$. Rearranging this equality, we see that G preserves the form with matrix Φ. $\qquad\square$

The procedure `PrimesForIsometry` accepts an absolutely irreducible subgroup H of $\mathrm{SL}(n,\mathbb{Z})$ that is not an isometry group. It selects $h \in H$ such that $a := \mathrm{tr}(h) - \mathrm{tr}(h^{-1}) \neq 0$, and (using [26]) returns those $p \in \pi(a)$ such that $\varphi_p(H)$ is an isometry group.

We will need to check not only whether a congruence image of H preserves a form, but whether it lies in the similarity group generated by a full isometry group and all scalars. This is achieved with `PrimesForSimilarity`(H), which selects $h = [h_1, h_2] \in H$ such that $a := \mathrm{tr}(h) - \mathrm{tr}(h^{-1}) \neq 0$. Clearly $\varphi_p(H)$ is in a similarity group only if $p \in \pi(a)$.

Lemma 2.10. *Suppose that $H \leq \mathrm{SL}(n,\mathbb{Z})$ is absolutely irreducible and not an isometry group. Then for almost all primes p, $\varphi_p(H)$ does not lie in a similarity group over $\mathbb{Z}_p$.*

3. ALGORITHMS FOR STRONG APPROXIMATION

We proceed to formulate an algorithm that realizes strong approximation in prime degree n. That is, the algorithm computes $\Pi(H)$ for any dense input $H \leq \mathrm{SL}(n,\mathbb{Z})$. We also compute Π for dense subgroups of $\mathrm{SL}(2n,\mathbb{Z})$ containing a transvection.

3.1. **Density in prime degree.** For the entirety of this subsection, n is prime.

By [2] (cf [23, p. 397]), the set $\mathcal{C}$ of maximal subgroups of $\mathrm{SL}(n,p)$ is a union of certain subsets $\mathcal{C}_1, \ldots, \mathcal{C}_9$. For each i, we determine all primes p such that $\varphi_p(H)$ could be in a group in $\mathcal{C}_i$. Hence, we provide criteria for H to surject onto $\mathrm{SL}(n,p)$ for almost all primes p. These conditions turn out to be equivalent to density. They constitute the background of our main algorithm and obviate any need to test density of the input (as in, say, [7, Section 5]).

We start with an auxiliary statement for $\mathcal{C}_9$ (called class $\mathcal{S}$ in [3, Chapter 8]).

Lemma 3.1. *There is a bound in terms of n on the order of subgroups of $\mathrm{SL}(n,p)$ that are contained solely in groups in $\mathcal{C}_9$.*

Proof. Suppose that $U \leq \mathrm{SL}(n,p)$ lies only in $\mathcal{C}_9$ and not in $\mathcal{C}_i$ for $i \neq 9$. The perfect residuum U^∞ (i.e., the last term of the derived series of U) is therefore a simple absolutely irreducible subgroup of $\mathrm{SL}(n,p)$. If we show that the order of U^∞ is bounded, then $U \leq \mathrm{Aut}(U^\infty)$ also has bounded order. Thus, without loss of generality, $U = U^\infty$ from now on.

Prime degree faithful representations of quasisimple groups are classified in [25, Theorem 1.1]. The orders of the groups in classes (10)–(27) of this classification are bounded absolutely (i.e., by a bound not depending on n or p). The orders of groups in classes (2)–(9) are bounded by a function of n.

Class (1) groups are of Lie type in characteristic p in the Steinberg representation [16], whose degree n is the p-part of the group order. For each class of groups of Lie type $\mathcal{G}_m(p)$, this p-part is p^a with $a \leq 1$ for only finitely many values of m. So class (1) is finite for prime n.

Finally we come to the case excluded by [25, Theorem 1.1], namely $U/Z(U) \cong \mathrm{Alt}(m)$ for $m > 18$. As a consequence of [17, 19], there are only finitely many degrees l such that $\mathrm{Sym}(l)$ and thus $\mathrm{Alt}(l)$ has a faithful (projective) representation of degree m. $\qquad\square$

The main procedure, `PrimesForDense`(H), combines the subsidiary procedures of Section 2. Its output is the union of

- `PrimesForAbsIrreducible`(H)
- `PrimesForMonomial`(H)
- `PrimesForSolvable`(H, δ), where δ is a bound on the derived length of a solvable linear group of degree n
- `PrimesForSimilarity`(H)
- `PrimesForOrder`(H, k) where k is a bound on element orders for groups of degree n in $\mathcal{C}_6 \cup \mathcal{C}_9$.

Theorem 3.2. *Assuming termination for input H, `PrimesForDense`(H) returns $\Pi(H)$.*

Proof. Each prime returned must lie in $\Pi(H)$. Conversely, let p be a prime such that $\varphi_p(H) \neq \mathrm{SL}(n, p)$. Then $\varphi_p(H)$ is in a group in some $\mathcal{C}_i$, $1 \leq i \leq 9$. For each i, we show that (at least) one of the subsidiary procedures returns p.

$\mathcal{C}_1$: here $\varphi_p(H)$ is reducible, so p is returned by `PrimesForAbsIrreducible`(H).

$\mathcal{C}_2$: p is returned by `PrimesForMonomial`(H).

$\mathcal{C}_3$: for prime n, the stabilizers of extension fields are solvable, so p is returned by `PrimesForSolvable`(H, δ).

$\mathcal{C}_4, \mathcal{C}_7$: since the degree of a tensor product is the product of the factor degrees, and n is prime, these classes are empty.

$\mathcal{C}_5$ is empty over fields of prime size.

$\mathcal{C}_6$ consists of groups whose structure depends on n but not on p [3, Section 2.2.6]. The number of such groups (and thus the largest order of an element in any one of them) is bounded, and so `PrimesForOrder`(H, k) returns p.

$\mathcal{C}_8$: the groups in this class preserve a form modulo $Z(\mathrm{SL}(n, p))$. Hence the derived group of $\varphi_p(H)$ preserves a form and p is returned by `PrimesForSimilarity`(H).

$\mathcal{C}_9$: by Proposition 3.1, the number of groups in this class is finite. Thus (as with $\mathcal{C}_6$) `PrimesForOrder`(H, k) returns p. $\qquad\square$

Remark 3.3. Using GAP and tables in [3, Chapter 8], we can calculate bounds on the order of groups in $\mathcal{C}_6 \cup \mathcal{C}_9$ (and hence bounds on their element orders) for small n. For $n = 2, 3, 5, 7, 11$, these bounds are $10, 21, 60, 84, 253$, respectively.

Remark 3.4. `PrimesForDense` simplifies in small degrees. If $n \leq 3$ then the groups in $\mathcal{C}_2$ are solvable, so `PrimesForSolvable` overrides `PrimesForMonomial`. In degree 2, `PrimesForSimilarity` is also redundant.

If `PrimesForDense`(H) terminates then $\Pi(H)$ is finite, i.e., H is dense [27, p. 3650]. Next we prove the converse. This leads to a characterization of density in $\mathrm{SL}(n, \mathbb{Z})$.

Lemma 3.5. *If H is irreducible, not solvable-by-finite, and not an isometry group, then $\Pi(H)$ is finite.*

Proof. Each constituent output set is finite by Lemmas 2.2, 2.3, 2.6, 2.10, and 3.1. $\qquad\square$

Lemma 3.6. *If H is infinite, non-solvable, primitive, and not an isometry group, then $\Pi(H)$ is finite.*

Proof. As the previous proof, but relying on Corollary 2.5 instead of Lemma 2.3. □

Lemma 3.7. *Suppose that $\varphi_p(H) = \mathrm{SL}(n, p)$ for some prime p, where $p > 3$ if $n = 2$. Then H is infinite, non-solvable, and primitive. Furthermore, H is not an isometry group.*

Proof. Since $\mathrm{SL}(n, p)$ is absolutely irreducible and non-solvable, the same is true of H. A monomial subgroup of $\mathrm{SL}(n, \mathbb{Z})$ cannot surject onto $\mathrm{SL}(n, p)$ because it has an abelian normal subgroup whose index is too small. The remaining assertion follows from Lemmas 2.1 and 2.9. □

Lemmas 3.6 and 3.7 yield

Corollary 3.8. *If $\varphi_q(H) = \mathrm{SL}(n, q)$ for one prime $q > 3$, then $\varphi_p(H) = \mathrm{SL}(n, p)$ for almost all primes p.*

Remark 3.9. Corollary 3.8 should be compared with [23, p. 396], [24, Proposition 1], and [33].

Corollary 3.10. *The following are equivalent.*

   (i) *H is dense.*
  (ii) *H surjects onto $\mathrm{SL}(n, p)$ modulo some prime p, where $p > 3$ if $n = 2$.*
 (iii) *H is infinite, non-solvable, primitive, and not an isometry group.*
 (iv) *H is irreducible, not solvable-by-finite, and not an isometry group.*

Remark 3.11. Let $n = 2$. Then H is dense if and only if H is not solvable-by-finite; which is equivalent to H being infinite and non-solvable.

To round out the subsection, we give one more set of criteria for density in odd prime degree.

Lemma 3.12. *Let $n > 2$. If H contains an irreducible element and is not solvable-by-finite then H is dense.*

Proof. We appeal to Lemma 3.5. Let $h \in H$ be irreducible. Suppose that H preserves a form with (symmetric or skew-symmetric) matrix Φ. Then $x \mapsto \Phi x^\top \Phi^{-1}$ defines a $\mathbb{Q}$-automorphism of $\langle h \rangle_{\mathbb{Q}}$ of order 2. But $\langle h \rangle_{\mathbb{Q}}$ is a field extension of odd degree n. Hence H is not an isometry group. □

Corollary 3.13. *For $n > 2$, a finitely generated subgroup of $\mathrm{SL}(n, \mathbb{Z})$ is dense if and only if it contains an irreducible element and is not solvable-by-finite.*

Remark 3.14. Lemma 3.6 allows us to replace 'not solvable-by-finite' in Lemma 3.12 and Corollary 3.13 by 'infinite non-solvable primitive', or by 'infinite non-solvable' if $n = 3$ (cf. [20, p. 415], [21, Theorem 2.2]).

3.2. **Algorithms for groups with a transvection.** In [7, Section 3.2] we gave a straight-forward procedure `PrimesForDense` to compute $\Pi(H)$ if H is dense in $\mathrm{SL}(2n+1, \mathbb{Z})$ or $\mathrm{Sp}(2n, \mathbb{Z})$ and contains a known transvection. The case $H \leq \mathrm{SL}(2n, \mathbb{Z})$ was left open. Now we close that gap.

Lemma 3.15. *Suppose that $H \leq \mathrm{SL}(2n, \mathbb{Z})$ contains a transvection t. Then H is dense if and only if $N := \langle t \rangle^H$ is absolutely irreducible and $\mathrm{tr}(h) \neq \mathrm{tr}(h^{-1})$ for some h in N.*

Proof. Suppose that H is dense. Then N is absolutely irreducible by [7, Corollary 3.5]. If $\mathrm{tr}(h) = \mathrm{tr}(h^{-1})$ for all $h \in N$, then by Lemma 2.9 there is a form with matrix Φ such that $h\Phi h^\top = \Phi$. Since $N \trianglelefteq H$ and N is absolutely irreducible, $h\Phi h^\top = \alpha\Phi$ for all $h \in H$ and some $\alpha \in \mathbb{Q}$ (see, e.g., [3, Lemma 1.8.9, p. 41]). This contradicts density of H.

Now suppose that N is absolutely irreducible and $\mathrm{tr}(h) \neq \mathrm{tr}(h^{-1})$ for some $h \in N$. Then $\varphi_p(N)$ is absolutely irreducible and $\varphi_p(\mathrm{tr}(h)) \neq \varphi_p(\mathrm{tr}(h^{-1}))$ for almost all primes p. So there are $p > 3$ and $g \in \varphi_p(N)$ such that $\varphi_p(N)$ is absolutely irreducible and $\mathrm{tr}(g) \neq \mathrm{tr}(g^{-1})$. Since $\varphi_p(N)$ is generated by transvections, the theorem of [34, p. 1] implies that $\varphi_p(N) = \mathrm{SL}(2n, p)$ or $\mathrm{Sp}(2n, p)$. Since the latter possibility is ruled out by Lemma 2.9, we must have $\varphi_p(H) = \mathrm{SL}(2n, p)$ and so H is dense (see [24, Proposition 1]). $\qquad\square$

The procedure `PrimesForDense`(H, t), based on Lemma 3.15, accepts dense $H \leq \mathrm{SL}(2n, \mathbb{Z})$ containing a transvection t, and returns $\Pi(H)$. It combines `PrimesForAbs-Irreducible`(N) and `PrimesForIsometry`(N), checking whether $\varphi_p(H) = \mathrm{SL}(2n, p)$ for each p in the union of their outputs. See [7, Section 3] for an algorithm to compute a basis of $\langle N \rangle_\mathbb{Q}$ without computing (a full generating set of) the normal closure N. Similarly, the application of `PrimesForIsometry` does not require computing N, and just randomly selects $h \in N$ such that $\mathrm{tr}(h) \neq \mathrm{tr}(h^{-1})$.

3.3. **General considerations.** We comment further on the operation of our algorithms.

When selecting (pseudo-)random elements of $\mathrm{SL}(n, \mathbb{Z})$ for some subprocedures, we seek just one element with a nominated property. These will be plentiful in dense subgroups. Hence we do not aim for any semblance of a uniform distribution (cf. [28, Section 5]), but randomly take words of length 5 in the given generators. If these repeatedly fail to have the desired property then we gradually increment the word length. We do not have a theoretical bound on the runtime for this process; but in practice we observe that it is very fast.

At the start of the calculation we also select (e.g., by computing the orders, or invoking composition tree on images of H modulo different primes [26]) a prime $p_0 > 3$ such that $\varphi_{p_0}(H) = \mathrm{SL}(n, p_0)$. The properties of elements that we are seeking may then be maintained modulo p_0. That is, instead of searching in H, we search for an element $\bar{h}$ in $\varphi_{p_0}(H)$ that has the desired properties (over $\mathbb{Z}_{p_0}$) and lift to the pre-image $h \in H$.

Each of the subsidiary procedures for `PrimesForDense`(H) returns a positive integer d divisible by every prime p such that $\varphi_p(H)$ is in the relevant class of maximal subgroups of $\mathrm{SL}(n, p)$. However, d can have prime factors not in $\Pi(H)$. Furthermore, these factors might be so large as to make factorization of d impractical, or make the test of the congruence image overly expensive. Thus we do not factor d fully, but only attempt a cheap partial

factorization (e.g., by trial division and a Pollard-ρ algorithm). If d does not factorize, or has large prime factors (magnitudes larger than the entries of the input matrices), then we compute another positive integer d' using the same algorithm but with different choices of random elements, and replace d by $\gcd(d, d')$.

Our computational realization of strong approximation stands in contrast to Breuillard's quantitative version [4, Theorem 2.3]. His bound on the primes that can appear in $\Pi(H)$ is not explicit. We compute all of $\Pi(H)$; and can do so quickly, as shown in the next section.

4. EXPERIMENTING WITH LOW-DIMENSIONAL DENSE SUBGROUPS

In this section we present experimental results obtained from our GAP implementation of the algorithms. We demonstrate the practicality of our software and how it can be used to obtain important information about groups. In particular we describe all congruence images, as explained in the next subsection.

4.1. **Computing all congruence images.** Let $H \leq \mathrm{SL}(n, \mathbb{Z})$ be dense. As in [7, Section 2.4.1], let $\tilde{\Pi}(H) = \Pi(H) \cup \{2\}$ if $\varphi_2(H) = \mathrm{SL}(n, 2)$ and $\varphi_4(H) \neq \mathrm{SL}(n, 4)$; let $\tilde{\Pi}(H) = \Pi(H)$ otherwise. The disparity between $\tilde{\Pi}(H)$ and $\Pi(H)$ can arise only when $n \leq 4$, and $M(H)$ is even but $2 \notin \Pi(H)$. By [7, Theorem 2.18], $\tilde{\Pi}(H) = \pi(M(H))$. If $n > 2$ then $\varphi_k(H) = \varphi_k(\mathrm{cl}(H))$ for all k; so $\tilde{\Pi}(H) = \tilde{\Pi}(\mathrm{cl}(H))$. We may therefore assume that H is arithmetic, of level M. Let $a = \gcd(k, M)$, so $k = abc$, $\pi(b) \subseteq \pi(a)$, and $\gcd(c, a) = 1$. Then $\varphi_k(H) \cong H/(H \cap \Gamma_k) \cong H\Gamma_k/\Gamma_k$ is a subgroup of $\Gamma_{ab}/\Gamma_k \times \Gamma_c/\Gamma_k$. It is not difficult to show that $\varphi_k(H)$ splits as a direct product of Γ_{ab}/Γ_k with $Q := ((H\Gamma_k) \cap \Gamma_c)/\Gamma_k$. Since $\Gamma_{ab}/\Gamma_k \cong \mathrm{SL}(n, \mathbb{Z})/\Gamma_c$, this expresses $\varphi_k(H)$ as a direct product of Q with a subgroup isomorphic to $\mathrm{SL}(n, \mathbb{Z}_c)$. Hence the task in describing all congruence images of H boils down to computing with the quotient Q of $\varphi_k(H)$ in $\mathrm{SL}(n, \mathbb{Z}_k)$; in effect, ranging over all divisors of M. If $n = 2$ then the congruence subgroup property does not hold, and we can only handle $k = p$ prime.

In some of the examples below we describe the congruence quotient modulo the level M, exhibiting which parts of its structure arise for various prime powers. We give this as an ATLAS-style composition structure [5] (separating composition factors by dots; cf. [3]), marked up to show the prime powers for which each factor first arises. We emphasize that these have been generated 'semiautomatically' using *some* composition series that refines the congruence structure, not necessarily the best possible series. One example from Table 1 is a group of level $3^4 5 \cdot 19$ with quotient structure

$$\underset{3^4}{3^4}.\underset{3^3}{3^3}.\underset{3^2}{3^3}.\underset{5}{5^2}.\underset{}{2.2.2}.\underset{3,5}{3.3}.\underset{19}{L_2(19)}$$

In the standard notation $L_m(q) := \mathrm{PSL}(m, q)$, this has congruence image $L_2(19)$ modulo 19, which is a simple direct factor not interacting with the other primes. The quotient modulo 3 has structure 3.3 (and is almost certainly the group 3^2). The quotient modulo 5 is $5^2.2.2.2.3.3$, forming a subdirect product with the quotient of order 3 in which the full factor 3.3 is glued together. Modulo 9 the group possesses a factor 3^3 (of the possible $3^{3 \cdot 3 - 1} = 3^8$), modulo 27 another factor 3^3, and modulo 81 a factor 3^4. (Since 3^4 is the

prime power dividing the level, the quotient modulo 243 would contain a full 3^8.) The structural analysis in [7, Section 2] proves that the exponent for p^{i+1} cannot be smaller than the exponent for p^i. The name indicates all proper prime powers dividing the level. Thus 'empty' factors $\underset{p^a}{\_}$ are possible if the group has no elements on that level.

Experimental results are displayed in Tables 1 and 2 (writing A_m for $\mathrm{Alt}(m)$ and S_m for $\mathrm{Sym}(m)$). Our actual implementation computes $\tilde{\Pi} = \pi(M)$ rather than $\Pi(H)$. We do not state $\Pi(H)$; as noted above, this set almost always coincides with $\pi(M)$.

Experiments were performed on a 2013 MacPro with a 3.7 GHz Intel Xeon E5 utilizing up to 8GB of memory. The software can be accessed at `http://www.math.colostate.edu/~hulpke/arithmetic.g`. Some documentation [8] is also available.

4.2. **Low-dimensional dense subgroups.** Our examples in this subsection come from a family of integral representations of finitely presented groups, as defined in [20, 21, 22]. For each test group H we compute $\Pi(H)$, incidentally justifying that H is dense. Thereafter we compute $M(H)$, $|\mathrm{SL}(n, \mathbb{Z}) : H|$, and the congruence quotients of H.

4.2.1. *The fundamental group of the figure-eight knot complement.* Adopting the notation of [20, p. 414], let

$$\Gamma := \langle x, y, z \mid zxz^{-1} = xy, zyz^{-1} = yxy \rangle;$$

this is the fundamental group of the figure-eight knot complement. Put $F = \langle x, y \rangle$. In [20], two families of representations β_T, ρ_k of Γ in $\mathrm{SL}(3, \mathbb{Z})$ were constructed. Section 4 of [7] reports on experiments with β_T for a range of T and ρ_k for $k = 0, 2, 3, 4, 5$. The groups $\rho_k(\Gamma)$, $\rho_k(F)$ for $k \neq 0, 2, 3, 4, 5$ are of special interest (see [20, Section 5]). However, neither the methods of [20] nor those of [7] facilitate proper study of ρ_k for such k.

We have

$$\rho_k(x) = \begin{pmatrix} 1 & -2 & 3 \\ 0 & k & -1-2k \\ 0 & 1 & -2 \end{pmatrix}, \quad \rho_k(y) = \begin{pmatrix} -2-k & -1 & 1 \\ -2-k & -2 & 3 \\ -1 & -1 & 2 \end{pmatrix},$$

$$\rho_k(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & -1-k \end{pmatrix}.$$

The results of experiments with $\rho_k(\Gamma)$ and $\rho_k(F)$ are collected in Table 1; here M is the level (which turns out to be the same for both Γ and F), Index_Γ is $|\mathrm{SL}(3, \mathbb{Z}) : \mathrm{cl}(\rho_k(\Gamma))|$, and $\mathrm{Index}_{\Gamma,F}$ is $|\mathrm{cl}(\rho_k(\Gamma)) : \mathrm{cl}(\rho_k(F))|$. The last column is the congruence image of $\rho_k(F)$ modulo M. For $k = 1, 6, 10$, the groups surject modulo 2 but not modulo 4.

Determination of the exceptional primes was instantaneous. The time to calculate level and index increased roughly with level, from a few seconds for $k = 1$ to about 15 minutes for $k = 20$.

k	M	Index$_\Gamma$	Index$_{\Gamma,F}$	StructureF
1	$2^2 3^4$	$2^{10}3^{15}13$	2^2	$\underbrace{3^4}.\underbrace{3^3}.\underbrace{}_{2^2}.\underbrace{3^3}_{3^2}.\underbrace{3}_{3}.3.\underbrace{L_3(2)}_{2}$
6	$2^2 31\cdot 43$	$2^{10}3^3 7\cdot 43^2 331\cdot 631$	$2\cdot 3\cdot 5$	$\underbrace{}_{2^2}.31.31.2.\underbrace{L_2(43)}_{43}.\underbrace{L_2(31)}_{31}.\underbrace{L_3(2)}_{2}$
7	$3^4 5\cdot 19$	$2^6 3^{17}5\cdot 13\cdot 19^2 31\cdot 127$	$2^2 3^2$	$\underbrace{3^4}.\underbrace{3^3}.\underbrace{3^3}_{3^2}.\underbrace{5^2}_{5}.2.2.2.\underbrace{3.3}_{3,5}.\underbrace{L_2(19)}_{19}$
10	$2^2 3^4 11\cdot 37$	$2^{14}3^{16}7^2 13\cdot 19\cdot 37^2 67$	$2^2 3^2 5$	$\underbrace{3^4}.\underbrace{3^3}.\underbrace{}_{2^2}.\underbrace{3^3}_{3^2}.\underbrace{11}_{11}.11.2.\underbrace{3.}_{3}\underbrace{3}_{3,11}$ $.\underbrace{L_2(37)}_{37}.\underbrace{L_2(11)}_{11}.\underbrace{L_3(2)}_{2}$
15	$229\cdot 241$	$2^6 3^3 5\cdot 97\cdot 181\cdot 241^2 19441$	$2\cdot 3\cdot 19$	$\underbrace{229.229.2}_{229}.\underbrace{L_2(241)}_{241}.\underbrace{L_2(229)}_{229}$
20	$409\cdot 421$	$2^4 3^3 5\cdot 7\cdot 421^2 55897\cdot 59221$	$2^2 3\cdot 17$	$\underbrace{409.409.2}_{409}.\underbrace{L_2(421)}_{421}.\underbrace{L_2(409)}_{409}$

<center>TABLE 1.</center>

4.2.2. *Triangle groups.* Next we look at triangle groups $\Delta(p,q,r) = \langle a,b \mid a^p = b^q = (ab)^r = 1\rangle$.

In [21], representations of $\Delta(3,3,4)$ in $\mathrm{SL}(3,\mathbb{Z})$ are defined by

$$a \mapsto a_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto b_1(t) = \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (-1+t)^2 \end{pmatrix}.$$

These representations are faithful for all $t \in \mathbb{R}$, and if $t \in \mathbb{Z}$ then the images are dense and non-conjugate for different t [21, Theorem 1.1]. If $t = 1$ then the group is conjugate to the one constructed by Kac and Vinberg [20, p. 422]. Put $H_1(t) = \langle a_1, b_1(t)\rangle$.

In [21, p. 8], the following faithful dense representations $H_2(t) = \langle a_2(t), b_2\rangle$ of $\Delta(3,4,4)$ were constructed:

$$a \mapsto a_2(t) = \begin{pmatrix} 1 & 4+3t^2/4 & 3(6-t+t^2)/2 \\ 0 & -(4+t+t^2)/2 & -3-t^2 \\ 0 & (4+2t+t^2)/4 & (2+t+t^2)/2 \end{pmatrix}, \quad b \mapsto b_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In [22, p. 13], faithful representations of $\Delta(3,3,4)$ in $\mathrm{SL}(5,\mathbb{Z})$ are defined by

$$a \mapsto a_3(k) = \begin{pmatrix} 1 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 4(-1+k) & -13-4k & 3+16(1+k)^2 & -4+16k \\ 0 & 1-k+4k^2 & 3-2k+8k^2 & -2(1+3k+16k^3) & 3+16k^2 \\ 0 & k & 2k & 1-2k-8k^2 & 1+4k \\ 0 & 0 & 3k & 3(-1+k-4k^2) & -2 \end{pmatrix},$$

$$b \mapsto b_3(k) = \begin{pmatrix} 0 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 1 & 3+4k & -13-8k-16k^2 & 4-16k \\ 0 & 0 & -2(1+k+4k^2) & 6k+32k^3 & -3-16k^2 \\ 1 & 0 & -2(1+k) & -1+2k+8k^2 & -1-4k \\ 2k & 0 & 1-2k & -4k & 1 \end{pmatrix}.$$

As k ranges over $\mathbb{Z}$, the $H_3(k) = \langle a_3(k), b_3(k)\rangle$ are dense and pairwise non-conjugate.

It is known that $H_1(t)$, $H_2(t)$, $H_3(k)$ are thin [21, 22]. For each of these groups we computed its level M and the index of its arithmetic closure in $\mathrm{SL}(n, \mathbb{Z})$ for several values of the parameters. See Table 2.

For $t \equiv 1 \pmod{4}$, the $H_1(t)$ as far as we tested surject onto $\mathrm{SL}(3, 2)$ but not onto $\mathrm{SL}(3, \mathbb{Z}_4)$.

Runtimes for degree 3 groups were consistent with the previous example. In degree 5, identification of primes was again instantaneous, while the calculation of level and index took about 6 minutes for $H_3(0)$ and 20 minutes for $H_3(3)$. So we did not try larger k.

4.2.3. *Random generators.* We constructed subgroups of $\mathrm{SL}(n, \mathbb{Z})$ for $n = 3, 5$ generated by a pair of pseudo-random matrices (via the GAP command RandomUnimodularMat). More than half of the groups so generated surject onto $\mathrm{SL}(n, p)$ modulo all primes p (and modulo 4). We attempted to verify whether each group is arithmetic by expressing its generators as words in standard generators of $\mathrm{SL}(n, \mathbb{Z})$ and running a coset enumeration with the presentation from [30]. As the enumeration never terminated, we suspect that these groups are not arithmetic (note that a random finitely generated subgroup of $\mathrm{SL}(n, \mathbb{Z})$ is likely to be thin [13, 27]).

4.2.4. *Further experimentation.* Comparing congruence images with finite quotients (obtained, e.g., by the low-index algorithm of [15, Section 5.4]) may help to decide whether a dense representation of a finitely presented group is faithful, or justify that a group is thin. For example, low-index calculations with the finitely presented group Γ as in Subsection 4.2.1 expose quotients (such as $\mathrm{Sym}(23)$, $\mathrm{Sym}(29)$, $\mathrm{Alt}(11) \wr C_2$, to name just a few) that cannot be congruence images of any $\rho_k(\Gamma)$, as they do not have representations of suitably small degree. Thus ρ_k cannot be faithful on Γ if $\rho_k(\Gamma)$ is arithmetic (cf. [20, Question 5.1]). This fact has a clear explanation: F is free and normal in Γ; hence a representation of Γ in $\mathrm{SL}(3, \mathbb{Z})$ is arithmetic precisely when its restriction on F is arithmetic [20, p. 420]; but any virtually free group cannot have a faithful arithmetic representation in $\mathrm{SL}(n, \mathbb{Z})$ for $n > 2$.

To illustrate another potential application of our algorithms, we show that faithful dense representations of the triangle groups $\Delta(3, 3, 4)$, $\Delta(3, 4, 4)$ in $\mathrm{SL}(3, \mathbb{Z})$ or $\mathrm{SL}(5, \mathbb{Z})$ are not arithmetic; this includes $H_1(t)$, $H_2(t)$, $H_3(k)$ as in Subsection 4.2.2 (cf. [21, 22]). Indeed, $\Delta(3, 3, 4)$ and $\Delta(3, 4, 4)$ each have a quotient isomorphic to $\mathrm{Alt}(20)$. This is not a congruence quotient of an arithmetic group in $\mathrm{SL}(3, \mathbb{Z})$ or $\mathrm{SL}(5, \mathbb{Z})$, because $\mathrm{Alt}(20)$ does not have a faithful representation in $\mathrm{SL}(3, p)$ or $\mathrm{SL}(5, p)$ for any p.

We also use this example to compare the capability of our algorithm with that of the low-index algorithm. Congruence quotients of $\rho_k(\Gamma)$ (modulo any integer $m > 1$, including m not dividing the level) produced by our algorithms expose quotients of Γ (such as $\mathrm{SL}(n, p)$ for large p) that are infeasible to find through a low-index computation, because these groups do not have a faithful permutation representation of sufficiently small degree. Using a homomorphism search [15, Section 9.1.1], we find that Γ has 34 normal subgroups N such that $\Gamma/N \cong \mathrm{SL}(3, 5)$. Applying our algorithm, we identify 80 values of k in the range $1, \ldots, 100$, such that $5 \notin \Pi(\rho_k(\Gamma))$. For these k, the kernels of the induced surjections

$\Gamma \to \rho_k(\Gamma) \to \mathrm{SL}(3,5)$ expose just 4 of the 34 normal subgroups. This prompts us to conjecture that the ρ_k will not expose all $\mathrm{SL}(n,p)$ quotients of Γ.

Group	Level	Index	Quotient
$H_1(1)$	$2^2 3 \cdot 5^2 19$	$2^{13}3^3 5^5 13 \cdot 19^3 31 \cdot 127$	$\underbrace{\ }_{2^2}.\underbrace{5^6}_{5^2}.3.2.2.\underbrace{3^2}_{19}.\underbrace{2}_{5,19}.\underbrace{2^2}_{3}.\underbrace{3}_{3,5}.\underbrace{A_6}_{19}.\underbrace{L_3(2)}_{2}$
$H_1(2)$	2^3	$2^7 7$	$\underbrace{2^5}_{2^3}.\underbrace{2^5}_{2^2}.\underbrace{2^2}_{2}.3$
$H_1(5)$	$2^3 7 \cdot 19 \cdot 31$	$2^{23}3^6 5^2 7^2 19^2 31^3 127 \cdot 331$	$\underbrace{\ }_{2^3}.\underbrace{\ }_{2^2}.\underbrace{3 \cdot 19^2}_{31}.\underbrace{2.2.2.}_{19}.\underbrace{3}_{7,19,31}.\underbrace{A_6}_{31}.\underbrace{L_2(7)}_{7}.\underbrace{L_3(2)}_{2}$
$H_1(9)$	$2^2 67$	$2^9 3^2 7^2 11^2 17 \cdot 31 \cdot 67$	$\underbrace{\ }_{2^2}.\underbrace{67^2}_{67}.2.2^2.3.\underbrace{L_3(2)}_{2}$
$H_1(10)$	$2^3 3 \cdot 7$	$2^{16}3^4 7^2 13 \cdot 19$	$\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{7.7}_{7}.\underbrace{3.3}_{3,7}.\underbrace{2.2^2}_{2,3,7}.3$
$H_1(12)$	$2^3 7 \cdot 31$	$2^{16}3^5 5^2 7^3 19 \cdot 31 \cdot 331$	$\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{31.31.2}_{31}.\underbrace{2^2}_{2,31}.\underbrace{3}_{2,7,31}.\underbrace{L_2(7)}_{7}$
$H_1(50)$	$2^3 601$	$2^{14}3^3 5^4 7^2 13 \cdot 43 \cdot 601 \cdot 9277$	$\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{601.601}_{601}.2.\underbrace{2^2}_{2,601}.3$
$H_1(100)$	$2^3 3 \cdot 19 \cdot 43$	$2^{19}3^9 5 \cdot 7^3 11 \cdot 13 \cdot 19 \cdot 43 \cdot 127 \cdot 631$	$\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{43.43}_{43}.\underbrace{19.19}_{19,43}.\underbrace{3.3.2.}_{3,19,43}.\underbrace{2^2}_{2,3,19,43}.3$
$H_2(2)$	$2^2 13$	$2^7 3^2 7 \cdot 61$	$\underbrace{2^2}_{2^2}.\underbrace{13.13}_{13}.\underbrace{2.2}_{2,13}.2^2.3.\underbrace{L_2(13)}_{13}.\underbrace{2}_{2,13}$
$H_2(10)$	$2^2 5 \cdot 109$	$2^{10}3^4 5^2 7^2 31 \cdot 571$	$\underbrace{2^2}_{2^2}.\underbrace{109.109.2.2.}_{109}.\underbrace{2^2}_{2,109}.\underbrace{3}_{2}.\underbrace{L_2(109)}_{109}.\underbrace{A_5}_{5}.\underbrace{2}_{2,5,109}$
$H_2(12)$	$2^8 3^4 17$	$2^{38}3^{15}7 \cdot 13 \cdot 307$	$\underbrace{2^6}_{2^8}.\underbrace{2^6}_{2^7}.\underbrace{2^5}_{2^6}.\underbrace{2^3}_{2^5}.\underbrace{2^3}_{2^4}.\underbrace{3^4}_{3^4}.\underbrace{2^2}_{2^3}.\underbrace{3^3.}_{3^3}\underbrace{.3^3}_{3^2}.\underbrace{17^2}_{17}.\underbrace{2.3.3.2.}_{3,17\ 2,3,17}.\underbrace{2^2}_{2,17}.\underbrace{3}_{2,3,17}.\underbrace{2}_{ }.\underbrace{L_2(17)}_{17}$
$H_2(50)$	$2^2 5^2 13 \cdot 193$	$2^{16}3^4 5^7 7^2 31 \cdot 61 \cdot 1783$	$\underbrace{2^2}_{2^2}.\underbrace{5^3}_{5^2}.\underbrace{193.193.2.13.13.2.2.}_{193}\underbrace{2^2}_{13,193}.\underbrace{.3}_{2,193}.\underbrace{L_2(193)}_{193}.\underbrace{L_2(13)}_{13}.\underbrace{A_5}_{5}.\underbrace{2}_{2,5,13,193}$
$H_3(0)$	$2^3 7^3 19^2$	$2^{52}3^{14}5^5 7^{32}19^{21}31 \cdot 127 \cdot 151 \cdot 181 \cdot 911 \cdot 2801$	$\underbrace{2^8}_{2^3}.\underbrace{7^1}_{7^3}8.\underbrace{2^6}_{2^2}.\underbrace{7^8}_{7^2}.\underbrace{19^{12}}_{19^2}.\underbrace{19^2}_{19}.2.2.2.\underbrace{2^6}_{2,19}.3.\underbrace{L_3(2)}_{2,7}$
$H_3(1)$	$2^6 67^2$	$2^{87}3^6 5^2 7^3 11^4 17^2 31^2 67^{20} \cdot 449 \cdot 761 \cdot 26881$	$\underbrace{2^{21}}_{2^6}.\underbrace{2^{11}}_{2^5}.\underbrace{2^8}_{2^4}.\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{67^{12}}_{67^2}.\underbrace{67.67}_{67}.2.\underbrace{2^2}_{2,67}.3$
$H_3(2)$	$2^3 13 \cdot 211^2$	$2^{33}3^9 5^6 7^5 13^5 31^2 37 \cdot 53^2 61 \cdot 113 \cdot 197$ $\cdot 211^{20} \cdot 1361 \cdot 30941 \cdot 292661$	$\underbrace{2^{16}}_{2^3}.\underbrace{2^{14}}_{2^2}.\underbrace{211^{12}}_{211^2}.\underbrace{211^2}_{211}.2.2.2.\underbrace{13.13.13.13.}_{13,211}\underbrace{2^6}_{2,13,211}.3.\underbrace{L_2(169)}_{13}.\underbrace{L_3(2)}_{2}$
$H_3(3)$	$2^6 7 \cdot 11^2 41^2$	$2^{106}3^8 5^{13}7^8 11^{20}19^2 \cdot 29^2 31 \cdot 41^{20}61$ $\cdot 1723 \cdot 2801 \cdot 3221 \cdot 579281$	$\underbrace{2^{21}}_{2^6}.\underbrace{2^{11}}_{2^5}.\underbrace{2^8}_{2^4}.\underbrace{2^5}_{2^3}.\underbrace{2^4}_{2^2}.\underbrace{11^{12}}_{11^2}.\underbrace{41^{12}}_{41^2}.\underbrace{41.41}_{41}.\underbrace{11.11.2.}_{11,41}$ $\underbrace{7.7.7.7.2.}_{7,11,41}\underbrace{2^2}_{2,7,11,41}.3.\underbrace{L_2(7).L_2(7)}_{7}$

TABLE 2.

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