Peculiarities of nilpotent matrices

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A $n \times n$ matrix with entries in a field $\mathbb{F}$ is called nilpotent if it has the following equivalent properties.

- $A^k = 0$ for some positive integer $k$.
- $A^n = 0$.
- All eigenvalues of $A$ are equal to zero.
- The characteristic polynomial of $A$ is $x^n$.

**Example**

*Every strictly upper triangular matrix in $M_n(\mathbb{F})$ is nilpotent.*
Every nilpotent matrix is similar to a strictly upper triangular matrix.

A subspace of \( M_n(\mathbb{F}) \) in which every element is nilpotent can have dimension at most \( \frac{n(n-1)}{2} \). A nilpotent space of this maximum dimension is similar to \( \text{SUT}_n(\mathbb{F}) \) (Gerstenhaber, 1958).

It is not true that every space of nilpotent matrices is triangulable. The problem of characterizing irreducible spaces of nilpotent matrices over fields remains open.
What about a (skew-)symmetric nilpotent space?

The trace of the square of a symmetric or skew-symmetric matrix is ± the sum of the squares of the entries. Over a real field, the only nilpotent symmetric or skew-symmetric matrix is the zero matrix (same for complex Hermitian or skew-Hermitian matrices).
What about a (skew-)symmetric nilpotent space?

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**Theorem 1 (Meshulam and Radwan, 1998)**

The maximum possible dimension of a space of symmetric matrices in $M_n(\mathbb{C})$ is

$$m^2 \quad \text{if} \quad n = 2m$$

$$m^2 + m \quad \text{if} \quad n = 2m + 1.$$  

For skew-symmetric matrices, the maximum dimension is

$$m^2 - m \quad \text{if} \quad n = 2m$$

$$m^2 \quad \text{if} \quad n = 2m + 1.$$
The arguments of Meshulam and Radwan can be adapted to give the following theorem, in which the stated dimension bounds can be attained if char $\mathbb{F} \neq 2$ and $-1$ is a square in $\mathbb{F}$.

**Theorem 2**

Let $\mathbb{F}$ be a field with char $\mathbb{F} \neq 2$. Let $\mathcal{N}_S$ and $\mathcal{N}_A$ respectively denote nilpotent spaces of symmetric and skew-symmetric matrices in $M_n(\mathbb{F})$. Then

\[
\begin{align*}
\dim \mathcal{N}_S & \leq m^2 & \text{if } n = 2m \\
\dim \mathcal{N}_S & \leq m^2 + m & \text{if } n = 2m + 1 \\
\dim \mathcal{N}_A & \leq m^2 - m & \text{if } n = 2m \\
\dim \mathcal{N}_A & \leq m^2 & \text{if } n = 2m + 1
\end{align*}
\]
Spaces achieving these bounds

Suppose char$\mathbb{F} \neq 2$ and write $i$ for a square root of $-1$ in $\mathbb{F}$. For $n = 2m + 1$, write

$$Z = \begin{pmatrix} I_{m \times m} & I_{m \times m} & 0_{m \times 1} \\ iI_{m \times m} & -iI_{m \times m} & 0_{m \times 1} \\ 0_{1 \times m} & 0_{1 \times m} & 1 \end{pmatrix}.$$ 

Symmetric nilpotent space of dimension $m^2 + m$:

$$Z^{-1}S_n(\mathbb{F})Z = \left\{ \begin{pmatrix} A & B & E_{(m \times 1)} \\ C & A^T & F_{(m \times 1)} \\ -2F^T & -2E^T & D \end{pmatrix} : A \in M_m(\mathbb{F}), B, C \in S_m(\mathbb{F}), D \in \mathbb{F} \right\}.$$ 

The subspace $S_N$ with $A$ strictly upper triangular, $C = 0$ and $F = 0$ is nilpotent of dimension $m^2 + m$, and is similar to a symmetric space.
Spaces achieving these bounds

Suppose \( \text{char} \mathbb{F} \neq 2 \) and write \( i \) for a square root of \(-1\) in \( \mathbb{F} \). For \( n = 2m + 1 \), write

\[
Z = \begin{pmatrix}
I_{m \times m} & I_{m \times m} & 0_{m \times 1} \\
iI_{m \times m} & -iI_{m \times m} & 0_{m \times 1} \\
0_{1 \times m} & 0_{1 \times m} & 1
\end{pmatrix}.
\]

**Skew-symmetric** nilpotent space of dimension \( m^2 \):

\[
Z^{-1}A_n(\mathbb{F})Z = \left\{ \begin{pmatrix} A & B & E_{(m \times 1)} \\ C & -A^T & F_{(m \times 1)} \\ 2F^T & 2E^T & 0 \end{pmatrix} : A \in M_m(\mathbb{F}) \right\}.
\]

The subspace \( A_N \) with \( A \) strictly upper triangular, \( C = 0 \) and \( F = 0 \) is nilpotent of dimension \( m^2 \), and is similar to a skew-symmetric space.
Decomposition of a nilpotent space of maximum dimension

In the above construction we have \( S_N \cap A_N = 0 \) and

\[
S_N + A_N = \left\{ \begin{pmatrix} U_{m \times m} & X_{m \times (m+1)} \\ 0_{(m+1) \times m} & L_{(m+1) \times (m+1)} \end{pmatrix} \right\},
\]

where \( U \) is strictly upper triangular, \( L \) is strictly lower triangular (and \( X \) can be anything). Thus \( S_N \oplus A_N \) is a space of nilpotent matrices in \( M_n(\mathbb{F}) \) of maximum possible dimension \( \frac{n(n-1)}{2} \).

**Theorem 3**

Let \( \mathbb{F} \) be a field of characteristic different from 2, in which \(-1\) is a square. Then for every positive integer \( n \), \( M_n(\mathbb{F}) \) contains a subspace of nilpotent matrices of dimension \( \frac{n(n-1)}{2} \) which is the direct sum of a symmetric nilpotent space and a skew-symmetric nilpotent space.
The strange case of characteristic 2

The story so far: for a field $\mathbb{F}$ whose characteristic is not 2, the dimension of a nilpotent subspace of skew-symmetric elements cannot exceed $m^2 - m$ if $n = 2m$ is even, or $m^2$ if $n = 2m + 1$ is odd. These bounds are attainable if $-1$ is a square in $\mathbb{F}$.

The corresponding numbers for fields of characteristic 2, which are different at least for $n$ odd:

- at least equal to $m^2 - m$ in the even case $n = 2m$;
- at least $m^2 + m$ in the odd case $n = 2m + 1$. 
Now suppose that $\text{char } \mathbb{F} = 2$. For $t = 1, \ldots, 5$, let $B_t$ be the element of $A_5(\mathbb{F})$ whose entries for $i \neq j$ are given by

$$(B_t)_{ij} = \begin{cases} 
1 & \text{if } t \in \{i, j\} \\
0 & \text{if } t \not\in \{i, j\}
\end{cases}$$

So for example

$$B_1 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
A special nilpotent space in $A_5(\mathbb{F})$

Now suppose that $\text{char} \mathbb{F} = 2$. For $t = 1, \ldots, 5$, let $B_t$ be the element of $A_5(\mathbb{F})$ whose entries for $i \neq j$ are given by

$$(B_t)_{ij} = \begin{cases} 
1 & \text{if } t \in \{i, j\} \\
0 & \text{if } t \notin \{i, j\}
\end{cases}$$

Let $R_5 = \langle B_1, B_2, B_3, B_4, B_5 \rangle$, a subspace of $A_5(\mathbb{F})$ of dimension 4.

$$R_5 = \left\{ \begin{pmatrix} 0 & a + b & a + c & a + d & a \\
a + b & 0 & b + c & b + d & b \\
a + c & b + c & 0 & c + d & c \\
a + d & b + d & c + d & 0 & c \\
a & b & c & d & 0 \end{pmatrix} : a, b, c, d, e \in \mathbb{F} \right\}.$$
Peculiar properties of $R_5$

$$R_5 = \left\{ \begin{pmatrix} 0 & a+b & a+c & a+d & a \\ a+b & 0 & b+c & b+d & b \\ a+c & b+c & 0 & c+d & c \\ a+d & b+d & c+d & 0 & d \\ a & b & c & d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{F} \right\}.$$ 

- Every element of $R_5$ is nilpotent.
- For every element $A$ of $A_5$ and every $B \in R_5$, the matrices $A$ and $A + B$ have the same characteristic polynomial.
- In particular if $N$ is any nilpotent subspace of $A_5(\mathbb{F})$, then $N + R_5$ is also a nilpotent subspace.
A nilpotent space of dimension $m^2 + m$ in $A_{2m+1}(\mathbb{F})$

- For any odd $n = 2m + 1$ we can define $R_n \subseteq A_n(\mathbb{F})$ as above.
- For odd $k \leq n$ think of $A_k(\mathbb{F})$ as the subspace of $A_n(\mathbb{F})$ consisting of those elements with zeros outside the upper left $k \times k$ region, so we can view $R_k(\mathbb{F})$ as a subspace of $M_n(\mathbb{F})$.
- Then $AN_n := R_3 + R_5 + \cdots + R_n$ is an irreducible nilpotent subspace of $A_n(\mathbb{F})$ of dimension $2 + 4 + \cdots + 2m = m^2 + m$. 
Example: \( n = 5, \ m = 2 \)

\[ AN_5(\mathbb{F}) \text{ consists of all matrices of the form} \]

\[
\begin{pmatrix}
0 & a + b + c + d & a + c + e & c + f & c \\
 a + b + c + d & 0 & b + d + e & d + f & d \\
 a + c + e & b + d + e & 0 & e + f & e \\
 c + f & d + f & e + f & 0 & f \\
 c & d & e & f & 0 \\
\end{pmatrix}.
\]