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# Algorithms for linear groups of finite rank

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## ABSTRACT

Let  $G$  be a finitely generated solvable-by-finite linear group. We present an algorithm to compute the torsion-free rank of  $G$  and a bound on the Prüfer rank of  $G$ . This yields in turn an algorithm to decide whether a finitely generated subgroup of  $G$  has finite index. The algorithms are implemented in MAGMA for groups over algebraic number fields.

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In [7,8] we developed practical methods for computing with linear groups over an infinite field  $\mathbb{F}$ . Those methods were used to test whether a finitely generated subgroup of  $\mathrm{GL}(n, \mathbb{F})$  is solvable-by-finite (SF). We now proceed to the design of further algorithms for finitely generated SF linear groups. Such a group may not be finitely presentable (see [21, 4.22, p. 66]), so obviously cannot be studied using approaches that require a presentation; in contrast to, say, polycyclic-by-finite (PF) groups. Extra restrictions are necessary to make computing feasible. Groups of finite rank are suitable candidates from this point of view, because they are well-behaved algorithmically [13, Section 9.3]. They also have convenient structural features (see [13, Section 5.2] and Section 1).

In this paper we develop initial results to enable computing with finitely generated linear groups of finite rank. Since such groups are  $\mathbb{Q}$ -linear (Proposition 1.4), our primary focus is the case that  $\mathbb{F}$  is an algebraic number field. We first test whether  $G \leq \mathrm{GL}(n, \mathbb{F})$  has finite rank. If so, we compute its torsion-free rank and an upper bound on its Prüfer rank. This furnishes an algorithm to decide whether a finitely generated subgroup of  $G$  has finite index. We determine various asymptotic bounds of interest in their own right. Algorithms for the structural investigation of  $G$  are provided as well: these construct a completely reducible part, and a finitely generated subgroup with the same rank as the unipotent radical. Our algorithms have been implemented in MAGMA [5]. We emphasize that

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computations are performed with a given group in its original representation, avoiding enlargement of matrices to get an isomorphic copy over  $\mathbb{Q}$ .

Naturally, it is possible to take advantage of additional properties of  $G$  when they are known. If  $G$  is polycyclic then one could obtain its torsion-free rank from a consistent polycyclic presentation of  $G$ , the latter found as in [2]. An even more tractable class is nilpotent-by-finite groups (cf. [10, Section 7]).

We summarize the layout of the paper. Section 1 gives background on linear groups of finite rank, including a reduction to SF groups over a number field. Section 2 is an extended treatment of such groups. In Section 3 we discuss ranks of finite index subgroups; we are indebted to D.J.S. Robinson for a vital theorem here. Section 3 also shows how to find the rank of a unipotent normal subgroup. In Section 4 we present our algorithms and some experimental results.

Unless stated otherwise,  $\mathbb{F}$  is an (infinite) field. The rational field is denoted as usual by  $\mathbb{Q}$ , and  $\mathbb{P}$  is a number field with ring of integers  $\mathcal{O}_{\mathbb{P}}$ .

## 1. Preliminaries

A general reference for this section is [13, Chapter 5].

### 1.1. Prüfer rank and torsion-free rank

Recall that a group  $G$  has finite Prüfer rank  $\text{rk}(G)$  if each finitely generated subgroup of  $G$  can be generated by  $\text{rk}(G)$  elements, and  $\text{rk}(G)$  is the least such integer.

**Theorem 1.1.** *Let  $G \leq \text{GL}(n, \mathbb{F})$  have finite Prüfer rank. Then  $G$  is SF, and if  $\text{char } \mathbb{F} > 0$  then  $G$  is abelian-by-finite (AF).*

**Proof.** See [21, 10.9, p. 141].  $\square$

**Corollary 1.2.** *Let  $G$  be a finitely generated subgroup of  $\text{GL}(n, \mathbb{F})$ . If  $G$  is AF then it has finite Prüfer rank; if  $G$  is completely reducible and has finite Prüfer rank then it is AF.*

**Proof.** If  $G$  is AF then it has a normal finitely generated abelian subgroup  $A$  of finite index. Since  $A$  and  $G/A$  have finite rank, so too does  $G$ . On the other hand, if  $G$  is completely reducible and has finite rank, then it is AF by Theorem 1.1 and [21, 3.5(ii), p. 44].  $\square$

**Remark 1.3.** The converse of Theorem 1.1 is not true even when  $G$  is finitely generated. However, see Proposition 2.3.

**Proposition 1.4.** *If  $G$  is a finitely generated subgroup of  $\text{GL}(n, \mathbb{F})$  of finite Prüfer rank then  $G$  is  $\mathbb{Q}$ -linear, i.e., isomorphic to a subgroup of  $\text{GL}(d, \mathbb{Q})$  for some  $d$ .*

**Proof.** Suppose that  $\text{char } \mathbb{F} = 0$ . By [21, 4.8, p. 56],  $G$  is (torsion-free)-by-finite, and by Theorem 1.1,  $G$  is SF. Thus  $G$  contains a torsion-free solvable normal subgroup of finite index and finite rank. The result now follows from [11, Theorem 2].

Suppose that  $\text{char } \mathbb{F} > 0$ . By Theorem 1.1,  $G$  is PF. It is well-known that a PF group is  $\mathbb{Z}$ -linear; see [13, 3.3.1, p. 57].  $\square$

Theorem 1.1 and Proposition 1.4 essentially reduce the investigation of finitely generated linear groups of finite rank to the case of SF groups over  $\mathbb{Q}$ . In Section 2.2 we show conversely that finitely generated SF subgroups of  $\text{GL}(n, \mathbb{P})$  always have finite rank. Hence we restrict attention mainly to groups over number fields.

Now recall that a group  $G$  has finite torsion-free rank if it has a subnormal series of finite length whose factors are either periodic or infinite cyclic. The number  $h(G)$  of infinite cyclic factors is the Hirsch number, or torsion-free rank, of  $G$ .

**Lemma 1.5.** *An SF group with finite Prüfer rank has finite torsion-free rank.*

**Proof.** See [13, p. 85].  $\square$

**Lemma 1.6.** *Let  $G$  be a group with normal subgroup  $N$ .*

- (i) *If  $G$  has finite Prüfer rank then  $\text{rk}(G) \leq \text{rk}(N) + \text{rk}(G/N)$ .*
- (ii) *If  $G$  has finite torsion-free rank then  $\text{h}(G) = \text{h}(N) + \text{h}(G/N)$ .*

## 1.2. Polyrational groups

Let  $U(G)$  be the unipotent radical of  $G \leq \text{GL}(n, \mathbb{F})$ ; namely, the largest unipotent normal subgroup of  $G$ . Note that  $G/U(G)$  is isomorphic to a completely reducible subgroup of  $\text{GL}(n, \mathbb{F})$ . If we exhibit  $G$  in block triangular form with completely reducible blocks, then  $U(G)$  is the kernel of the projection of  $G$  onto its main diagonal. Denote the largest periodic normal subgroup of  $G$  by  $\tau(G)$ .

**Lemma 1.7.** *Let  $G$  be a finitely generated subgroup of  $\text{GL}(n, \mathbb{F})$  of finite Prüfer rank. Then  $\tau(G)$  is finite.*

**Proof.** Theorem 1.1 and Proposition 1.4 imply that  $G$  is SF and we may assume that  $\text{char } \mathbb{F} = 0$ . Then  $\tau(G)$  is isomorphic to a subgroup of  $\tau(G/U(G))$ , and  $G/U(G)$  is finitely generated AF by Corollary 1.2. So we may further assume that  $G$  has a normal abelian subgroup  $A$  of finite index. Since  $A$  is finitely generated,  $\tau(G) \cap A \leq \tau(A)$  is finite. Thus  $|\tau(G)| = |\tau(G)A : A| \cdot |\tau(G) \cap A|$  is finite.  $\square$

A group is *polyrational* if it has a series of finite length with each factor isomorphic to a subgroup of the additive group  $\mathbb{Q}^+$ . So a polyrational group has finite torsion-free and Prüfer ranks.

**Proposition 1.8.** *If  $G$  is polyrational then  $\text{rk}(G) = \text{h}(G)$ .*

**Proof.** See [13, 5.2.7, p. 93].  $\square$

**Theorem 1.9.** *A finitely generated subgroup  $G$  of  $\text{GL}(n, \mathbb{F})$  has finite Prüfer rank if and only if it is polyrational-by-finite. In this case,  $\text{h}(G) \leq \text{rk}(G)$ .*

**Proof.** The first statement follows from Theorem 1.1, Lemmas 1.5 and 1.7, and [13, 5.2.5, p. 92]. For the second, let  $N$  be a normal polyrational finite index subgroup of  $G$ ; then  $\text{h}(G) = \text{h}(N) = \text{rk}(N) \leq \text{rk}(G)$ .  $\square$

Henceforth, the term ‘rank’ without a qualifier means Prüfer or torsion-free rank.

## 2. Solvable-by-finite groups over a number field

We now focus on finitely generated SF subgroups of  $\text{GL}(n, \mathbb{P})$ . Set  $|\mathbb{P} : \mathbb{Q}| = m$ . In this section we obtain more detailed information about these groups that will be used in our algorithms.

A finitely generated subgroup  $G$  of  $\text{GL}(n, \mathbb{F})$  is contained in  $\text{GL}(n, R)$  where  $R \subseteq \mathbb{F}$  is a finitely generated integral domain. The quotient ring  $R/\rho$  is a finite field for any maximal ideal  $\rho$  of  $R$ . We explain in [7, Section 2] how to construct a congruence homomorphism  $\varphi_\rho : \text{GL}(n, R) \rightarrow \text{GL}(n, R/\rho)$  for a maximal ideal  $\rho$  such that

- the kernel  $G_\rho$  of  $\varphi_\rho$  on  $G$  is unipotent-by-abelian (UA) if  $G$  is SF;
- $G_\rho$  is torsion-free if  $\text{char } \mathbb{F} = 0$ .

To be more explicit, let  $\mathbb{F} = \mathbb{P} = \mathbb{Q}(\alpha)$  where  $\alpha$  has minimal polynomial  $f(X)$ , and let  $G = \langle S \rangle$ . Then  $\varphi_\rho$  on  $R \cap \mathbb{Q}$  is reduction modulo an odd prime  $p \in \mathbb{Z}$  not dividing the discriminant of  $f(X)$  nor the denominators of entries in elements of  $S \cup S^{-1}$ . Hence  $\varphi_\rho$  maps  $R$  into the finite field  $\mathbb{Z}_p(\beta)$ , where  $\beta$  is a root of the mod  $p$  reduction of  $f(X)$ . We adhere to this notation from [7].

## 2.1. Unipotent groups

Denote the group  $\text{UT}(n, K)$  of upper unitriangular matrices over a commutative unital ring  $K$  by  $T$ . Define  $T_i$  to be the subgroup of  $T$  consisting of all matrices with their first  $i - 1$  superdiagonals equal to zero. Then  $T = T_1 > T_2 > \cdots > T_n = 1$  is the lower (and upper) central series of  $T$ . The homomorphism on  $T_i$  that maps each element to its  $i$ th superdiagonal has kernel  $T_{i+1}$  and image the  $(n - i)$ -fold direct sum  $K^+ \oplus \cdots \oplus K^+$ .

**Lemma 2.1.** *If  $G \leq \text{UT}(n, \mathbb{Q})$  then*

- (i)  *$G$  is polyrational,*
- (ii)  *$\text{rk}(G) = \text{h}(G) \leq n(n - 1)/2$ .*

**Proof.** Let  $K = \mathbb{Q}$  in the notation introduced just before the lemma. Since  $(G \cap T_i)/(G \cap T_{i+1})$  is isomorphic to a subgroup of  $T_i/T_{i+1}$ , (i) is clear. Then  $\text{rk}(G) = \text{h}(G)$  by Proposition 1.8. Also, by Lemma 1.6(ii),

$$\text{h}(T) = \text{h}(T_1/T_2) + \text{h}(T_2/T_3) + \cdots + \text{h}(T_{n-1}/T_n) = \sum_{i=1}^{n-1} i = n(n - 1)/2. \quad \square$$

**Corollary 2.2.** *If  $G \leq \text{UT}(n, \mathbb{P})$  then  $G$  is polyrational and  $\text{rk}(G) = \text{h}(G) \leq nm(nm - 1)/2$ .*

## 2.2. Ranks of solvable-by-finite groups over number fields

In this section  $G$  is a finitely generated subgroup of  $\text{GL}(n, \mathbb{P})$ . We prove that if  $G$  is SF then it has finite rank. Although  $\text{rk}(G)$  can be arbitrarily large, the ranks of finitely generated SF subgroups of  $\text{GL}(n, \mathcal{O}_{\mathbb{P}})$  are bounded by functions of  $n$  and  $m$ , which we give below.

**Proposition 2.3.** *Suppose that  $G$  is SF. Then  $G$  is polyrational-by-finite, hence of finite Prüfer rank.*

**Proof.** Select an ideal  $\rho$  such that  $G_\rho$  is UA and  $G/G_\rho$  is finite. Let  $U$  be the unipotent radical of  $G_\rho$ ; then  $G_\rho/U$  is finitely generated abelian. Write  $G_\rho/U = H/U \times \tau(G_\rho/U)$ . Since  $H/U$  is a finitely generated free abelian group and  $U$  is conjugate to a subgroup of  $\text{UT}(n, \mathbb{P})$ ,  $H$  is polyrational. Thus  $G_\rho$  has a polyrational normal subgroup of finite index. Consequently the same is true for  $G$ .  $\square$

**Remark 2.4.** Retaining the notation in the proof of Proposition 2.3,  $\text{h}(G) = \text{h}(G_\rho)$  and  $\text{rk}(G) \leq \text{rk}(G_\rho) + \text{rk}(\varphi_\rho(G))$  by Lemma 1.6. Furthermore  $\text{rk}(G_\rho) \leq \text{h}(H) + \text{rk}(\tau(G_\rho/U))$ . If we know  $x \in \text{GL}(n, \mathbb{P})$  that conjugates  $G$  to block upper triangular form with completely reducible diagonal blocks, then we can choose  $\rho$  so that the torsion-free group  $G_\rho$  is polyrational, and thus  $\text{rk}(G_\rho) = \text{h}(G_\rho)$ . In particular,  $G_\rho$  is polyrational for any  $\rho$  when  $G$  is completely reducible.

Remark 2.4 underpins our algorithm to calculate ranks.

**Corollary 2.5.** *A finitely generated subgroup of  $\text{GL}(n, \mathbb{F})$  has finite Prüfer rank if and only if it is SF and  $\mathbb{Q}$ -linear.*

**Proposition 2.6.** *The following are equivalent.*

- (i)  $G$  is SF.
- (ii)  $G$  has finite Prüfer rank.
- (iii)  $G$  has finite torsion-free rank.

**Proof.** Theorem 1.1 and Proposition 2.3 give (i)  $\Leftrightarrow$  (ii). Then (i)  $\Leftrightarrow$  (iii) by Lemma 1.5 and the Tits alternative.  $\square$

**Remark 2.7.** Thus, we can test whether  $G$  has finite rank using the algorithm of [7, Section 3.2], which decides the Tits alternative for  $G$ . This algorithm accepts a finitely generated linear group over any  $\mathbb{F}$ ; if it returns `false`, then the input does not have finite rank.

In fact, Proposition 2.3 holds for a wider class of groups: what is most important here is that unipotent subgroups of  $\mathrm{GL}(n, \mathbb{P})$  have finite rank.

**Lemma 2.8.** *If  $R$  is a finitely generated subring of  $\mathbb{P}$  then an SF subgroup  $H$  of  $\mathrm{GL}(n, R)$  has finite Prüfer rank.*

**Proof.** It suffices to confirm that  $H/U(H)$  has finite rank. Indeed,  $H/U(H)$  is finitely generated AF by [21, 4.10, p. 57].  $\square$

**Proposition 2.9.** *Suppose that  $G \leq \mathrm{GL}(n, \mathcal{O}_{\mathbb{P}})$  is SF. Then  $h(G) \leq nm(nm+1)/2$  and  $\mathrm{rk}(G) \leq nm(2nm+3)/2$ .*

**Proof.** Since  $\mathrm{GL}(n, \mathcal{O}_{\mathbb{P}})$  embeds into  $\mathrm{GL}(nm, \mathbb{Z})$ , we may assume without loss of generality that  $G \leq \mathrm{GL}(n, \mathbb{Z})$ .

(i) Suppose that  $G$  is abelian and  $\mathbb{Q}$ -irreducible. Then the enveloping algebra  $\langle G \rangle_{\mathbb{Q}}$  is a number field of degree  $n$  over  $\mathbb{Q}$ . Moreover,  $G$  is contained in the unit group of the ring of integers of  $\langle G \rangle_{\mathbb{Q}}$ . Hence  $\mathrm{rk}(G) \leq n$  by Dirichlet's Units Theorem [19, Theorem 12.6, p. 227].

(ii) If  $G$  is abelian and completely reducible over  $\mathbb{Q}$ , then [20, Lemma 4, p. 173] implies that  $G$  is conjugate to a group of block diagonal matrices  $\{\mathrm{diag}(\mu_1(g), \dots, \mu_k(g)) \mid g \in G\}$  where  $\mu_i(G) \leq \mathrm{GL}(n_i, \mathbb{Z})$  is  $\mathbb{Q}$ -irreducible. Therefore, by (i),

$$\mathrm{rk}(G) \leq \sum_{i=1}^k \mathrm{rk}(\mu_i(G)) = \sum_{i=1}^k n_i = n.$$

(iii) If  $G$  is UA then  $\mathrm{rk}(G) \leq \frac{n(n-1)}{2} + n = n(n+1)/2$  by (ii) and Lemma 2.1.

(iv) By Remark 2.4, there is an odd prime  $p$  such that  $h(G) = \mathrm{rk}(G_{\rho})$  and  $\mathrm{rk}(G) \leq \mathrm{rk}(G_{\rho}) + \mathrm{rk}(\varphi_{\rho}(G))$  for  $\rho = pR$ . Thus  $h(G) \leq n(n+1)/2$ . By [12], a finite completely reducible linear group of degree  $n$  can be generated by  $\lfloor 3n/2 \rfloor$  elements. Since  $\mathrm{rk}(\mathrm{UT}(n, p)) \leq n(n-1)/2$ , we deduce that  $\mathrm{rk}(\varphi_{\rho}(G)) \leq n(n+2)/2$ . The stated bound on  $\mathrm{rk}(G)$  follows.  $\square$

**Remark 2.10.** (i) If  $n \geq 4$  then the bound on  $\mathrm{rk}(G)$  in Proposition 2.9 can be improved using  $\mathrm{rk}(\mathrm{GL}(n, p)) \leq \frac{n^2}{4} + 1$ ; see [15, p. 199].

(ii)  $\mathrm{rk}(\mathrm{GL}(n, p)) \geq \lfloor n^2/4 \rfloor$  because  $\mathrm{UT}(n, p)$  has an elementary abelian subgroup of order  $p^{\lfloor n^2/4 \rfloor}$ .

### 3. Subgroups of finite index

In this section we first derive a rank-based criterion to recognize when a subgroup of a finitely generated linear group of finite rank has finite index. Subsequently we prove a result about the unipotent radical that forms a key piece of our main algorithm.

### 3.1. Ranks and isolators

We recall some definitions from [13, pp. 83–86]. The  $p$ -rank ( $p$  prime) of an abelian group is the cardinality of a maximal  $\mathbb{Z}_p$ -linearly independent subset of elements of order  $p$ . A solvable group  $G$  has *finite abelian ranks* ( $G$  is a *solvable FAR group*) if there is a series of finite length in  $G$  with each factor abelian, and of finite torsion-free rank and finite  $p$ -rank for every prime  $p$ . A *minimax group* is a group that has a series of finite length whose factors satisfy either the maximal condition or the minimal condition on subgroups. The minimality  $m(G)$  of a solvable minimax group  $G$  is the number of infinite factors in a series of  $G$  with each factor finite, cyclic, or quasicyclic. For finitely generated solvable groups, the notions of FAR, minimax, and finite Prüfer rank all coincide [13, pp. 175–176].

The following theorem and its proof were communicated to us by D.J.S. Robinson.

**Theorem 3.1** (D.J.S. Robinson). *Let  $H$  be a subgroup of a finitely generated solvable FAR group  $G$ . Then  $|G : H|$  is finite if and only if  $h(H) = h(G)$ .*

**Proof.** The ‘only if’ direction being clear, assume that  $h(H) = h(G)$ . For  $N \trianglelefteq G$ ,

$$h(HN/N) = h(H) - h(H \cap N) \geq h(G) - h(N) = h(G/N).$$

Thus  $h(HN/N) = h(G/N)$ . We prove that  $|G : H|$  is finite by induction on  $m(G)$ . If  $m(G) = 0$  then  $G$  is finite, so let  $m(G) > 0$ .

Denote the finite residual of  $G$  by  $D$ ; this is a divisible periodic abelian group [13, 5.3.1, p. 96]. Suppose that  $D \neq 1$ . Then  $m(G/D) < m(G)$ , and by the inductive hypothesis  $|G : HD|$  is finite. Hence  $HD$  is finitely generated, so  $HD = HD_0$  where  $D_0 \leq D$  is finitely generated, i.e., finite. This implies that  $|HD : H|$  is finite, as is  $|G : H|$ .

Suppose now that  $D = 1$ . Then  $G$  has a non-trivial torsion-free abelian normal subgroup  $A$  (for example, the penultimate term in the derived series of a non-trivial torsion-free normal subgroup of  $G$ ). Since  $m(G/A) < m(G)$ , by induction  $|G : HA|$  is finite. Next,  $H \cap A \neq 1$ ; otherwise  $h(H) = h(HA/A) = h(G/A) < h(G)$ . So the result holds for  $HA/(H \cap A)$  and its subgroup  $H/(H \cap A)$  by induction. Therefore  $|HA : H|$  is finite, as is  $|G : H|$ .  $\square$

**Remark 3.2.** Finitely generated linear groups are residually finite [21, 4.2, p. 51], so for our algorithms we only need that part of the proof of Theorem 3.1 in which  $D = 1$ .

**Corollary 3.3.** *Let  $H \leq G \leq \mathrm{GL}(n, \mathbb{F})$  where  $G$  is finitely generated and of finite Prüfer rank. Then  $|G : H|$  is finite if and only if  $h(H) = h(G)$ .*

The *isolator* in  $G$  of a subgroup  $H$  is

$$I_G(H) = \{x \in G \mid x^k \in H \text{ for some positive integer } k\}.$$

**Theorem 3.4.** *Let  $G$  be a finitely generated SF group, and let  $H \leq G$ . Then  $|G : H|$  is finite if and only if  $I_G(H) = G$ .*

**Proof.** See [13, 2.3.14, p. 45].  $\square$

**Lemma 3.5.** *Suppose that  $G$  is a solvable FAR group with a finitely generated subgroup  $H$  such that  $h(H) = h(G)$ . Then  $I_G(H) = G$ .*

**Proof.** Since  $h(\langle g, H \rangle) = h(H)$  for every  $g \in G$ , the lemma follows from Theorem 3.1.  $\square$



**Lemma 3.6.** Suppose that  $G$  is a group of finite torsion-free rank, and  $H$  is a subgroup of  $G$  such that  $I_G(H) = G$ . Then  $h(G) = h(H)$ .

We consider an illustrative example. Let  $G \leq \text{UT}(n, \mathbb{C})$  be an algebraic group defined over  $\mathbb{Q}$ , and set  $G_S := G \cap \text{GL}(n, S)$  for a subring  $S$  of  $\mathbb{C}$ . Recall that  $L \leq G_{\mathbb{Q}}$  is an arithmetic subgroup of  $G$  if  $L$  is commensurable with  $G_{\mathbb{Z}}$ ; i.e.,  $L \cap G_{\mathbb{Z}}$  has finite index in both  $L$  and  $G_{\mathbb{Z}}$ .

**Lemma 3.7.** A finitely generated subgroup  $L$  of  $G_{\mathbb{Q}}$  is an arithmetic subgroup of  $G$  if and only if  $\text{rk}(L) = \text{rk}(G_{\mathbb{Q}})$ .

**Proof.** By [17, Lemma 6, p. 138],  $H := L \cap G_{\mathbb{Z}}$  has finite index in  $L$ . Since  $L$  is polyrational and nilpotent,  $\text{rk}(H) = \text{rk}(L)$  by Theorem 3.1. Similarly (as  $G_{\mathbb{Z}}$  is finitely generated)  $|G_{\mathbb{Z}} : H| < \infty$  if and only if  $\text{rk}(G_{\mathbb{Z}}) = \text{rk}(H)$ . Also, it is not difficult to verify that  $G_{\mathbb{Q}} = I_{G_{\mathbb{Q}}}(G_{\mathbb{Z}})$ . Hence  $\text{rk}(G_{\mathbb{Q}}) = \text{rk}(G_{\mathbb{Z}})$  by Lemma 3.6.  $\square$

**Remark 3.8.** By Lemma 3.7 and [6, Corollary 7.2], if  $L$  is arithmetic in  $G$  then  $h(L)$  is the dimension of  $G$  as an algebraic group.

### 3.2. Prüfer rank of a unipotent normal subgroup

Let  $G$  be a finitely generated SF subgroup of  $\text{GL}(n, \mathbb{P})$ . We show how to construct a finitely generated subgroup of  $U(G)$  with the same Prüfer rank as  $U(G)$ .

Suppose that  $G = \langle x_1, \dots, x_r \rangle$ , and let  $Y$  be a finite subset of  $U(G)$ . The normal closure  $N = \langle Y \rangle^G$  is in  $U(G)$ . Define subgroups  $H_1 \leq H_2 \leq \dots$  of  $N$  as follows: let  $H_1 = \langle Y \rangle$ , and for  $i \geq 1$ , if  $H_i = \langle y_{i1}, \dots, y_{is_i} \rangle$  then

$$H_{i+1} = \langle y_{ij}, y_{ij}^{x_k}, y_{ij}^{x_k^{-1}} : 1 \leq j \leq s_i, 1 \leq k \leq r \rangle.$$

Since  $\text{rk}(H_i) \leq \text{rk}(H_{i+1}) \leq \text{rk}(N)$ , there exists  $t$  such that  $\text{rk}(H_t) = \text{rk}(H_{t+1})$ .

**Lemma 3.9.**  $\text{rk}(H_t) = \text{rk}(N)$ .

**Proof.** By Lemma 3.5,  $I_{H_{t+1}}(H_t) = H_{t+1}$ . So for  $1 \leq i \leq r$  and  $1 \leq j \leq s_t$ , there are positive integers  $m_{ij}, \bar{m}_{ij}$  such that  $(y_{tj}^{x_i})^{m_{ij}}, (y_{tj}^{x_i^{-1}})^{\bar{m}_{ij}} \in H_t$ . We claim that  $y_{tj}^x \in I_G(H_t)$  for all  $j$  and  $x \in G$ . First,

$$(y_{tj}^{x_v x_u^{\pm 1}})^{m_{vj}} = ((y_{tj}^{x_v})^{m_{vj}})^{x_u^{\pm 1}} \in H_{t+1}$$

since  $H_i^{x_k^{\pm 1}} \leq H_{i+1}$ . Similarly  $(y_{tj}^{x_v^{-1} x_u^{\pm 1}})^{\bar{m}_{vj}} \in H_{t+1}$ . Induction on the word length of  $x$  then establishes that  $y_{tj}^x \in I_G(H_t)$  as claimed. Hence  $N = H_1^G \leq H_t^G \subseteq I_G(H_t)$ ; i.e.,  $N = I_N(H_t)$ . By Lemma 3.6, the proof is complete.  $\square$

## 4. Computing ranks of solvable-by-finite linear groups

Let  $\mathcal{S}$  be a finite subset of  $\text{GL}(n, \mathbb{P})$  where  $|\mathbb{P} : \mathbb{Q}| = m$ , and let  $G = \langle \mathcal{S} \rangle$ . In this section we present algorithms to compute  $h(G)$  and a bound on  $\text{rk}(G)$ . These lead directly to an algorithm that tests whether a finitely generated subgroup of  $G$  has finite index.

Proposition 2.6 enables us first to test whether  $G$  has finite Prüfer (and thereby torsion-free) rank: `IsFiniteRank(G)` returns `true` precisely when the procedure `IsSolvableByFinite(G)` as in [7, p. 402] returns `true`. From now on,  $G$  has finite rank.



#### 4.1. Auxiliary procedures

4.1.1. Suppose that  $G$  is abelian and irreducible. Methods to construct a presentation of  $G$  are reasonably standard; see [1, Chapter 4] for details. We can find the homogeneous components of  $G$  (e.g., by [16]), so the methods extend to completely reducible abelian  $G$ . For such input we have procedures (i) `PresentationA`, which returns a presentation of  $G$ ; and (ii) `RankA`, which returns the torsion-free rank of  $G$ . Then  $\text{rk}(G) = \text{RankA}(G) + \varepsilon$  where  $\varepsilon = 0$  if  $G$  is torsion-free and  $\varepsilon = 1$  otherwise.

4.1.2. If  $G \leq \text{UT}(n, \mathbb{P})$  then  $G$  is isomorphic to a subgroup of  $\text{UT}(nm, \mathbb{Z})$  [17, Lemma 2, p. 111]. Since  $\text{UT}(nm, \mathbb{Z})$  is polycyclic, a constructive polycyclic sequence for  $G$  may be calculated as in [18, Chapter 9] or [1, Chapter 5]. From this one immediately reads off  $\text{RankU}(G) := h(G) = \text{rk}(G)$ .

#### 4.2. Completely reducible groups

If  $G$  is completely reducible then  $G_\rho$  is completely reducible abelian and  $h(G) = h(G_\rho)$ . Thus  $\text{RankCR}(G) := h(G) = \text{RankA}(G_\rho)$  as per Section 4.1.1.

Now let  $\mathbb{F}$  be arbitrary and  $G \leq \text{GL}(n, \mathbb{F})$  be finitely generated SF. In [7, Section 4] we show how to test whether  $G$  is completely reducible. Here we describe a more general procedure.

We refer to [7, Section 3.2]. The computations carried out in a run of `IsSolvableByFinite(G)` yield a change of basis matrix  $x$  such that  $G^x$  is block upper triangular and all diagonal blocks of  $G_\rho^x$  are abelian. Treating each diagonal block of  $G^x$  separately, assume that  $G_\rho$  is abelian. Let  $M = \{h_1, \dots, h_t\} = \text{NormalGenerators}(G_\rho)$ ; i.e.,  $G_\rho = \langle M \rangle^G$ . With a subscript ‘ $u$ ’ denoting unipotent part from a Jordan decomposition,  $H = \langle (h_1)_u, \dots, (h_t)_u \rangle = \langle M \rangle_u \leq (G_\rho)_u$ . Set  $U = \text{Fix}((G_\rho)_u)$  and  $W = \text{Fix}(H)$ . Since  $G$  normalizes  $(G_\rho)_u$ , we see that  $U$  is a  $G$ -module. We find  $U$  as follows.

- (1)  $\bar{W} := W$ .
- (2) While  $\exists g_i \in \mathcal{S}$  such that  $g_i \bar{W} \neq \bar{W}$   
 $\quad \bar{W} := g_i \bar{W} \cap W$ .
- (3) Return  $\bar{W}$ .

Clearly  $U \subseteq \bar{W}$ . Let  $v \in \bar{W}$  and  $g \in G$ ; then  $(h_i)_u^g v = g^{-1} (h_i)_u . g v = g^{-1} g v$  (because  $g v \in \bar{W} \subseteq W$ ) =  $v$ . This shows that  $\bar{W} = U$ . By [20, Theorem 5, p. 172],  $U$  is completely reducible as a  $G_\rho$ -module. Therefore, if  $\text{char } \mathbb{F}$  does not divide  $|G : G_\rho|$ , then  $U$  is a completely reducible  $G$ -module by [20, Theorem 1, p. 122]. Repeat the previous computation after replacing the current underlying space  $V$  for  $G$  by  $V/U$ . Continuing in this fashion, we eventually produce a flag  $V = V_1 \supset V_2 \supset \dots \supset V_l \supset \{0\}$  of  $G$ -modules with each quotient  $V_i/V_{i+1}$  completely reducible.

We adopt the following notation in our pseudocode. For a matrix group  $H$  in block upper triangular form,  $\mu$  denotes the projection of  $H$  onto its block diagonal, and  $\mu_i$  is the projection onto its  $i$ th diagonal block. When all diagonal blocks are completely reducible,  $\ker \mu = U(H)$  and  $\mu(H)$  is a ‘completely reducible part’ of  $H$ .

`CompletelyReduciblePart(G)`

Input: a finite subset  $\mathcal{S}$  of  $\text{GL}(n, \mathbb{F})$  such that  $\text{char } \mathbb{F}$  does not divide  $|G : G_\rho|$  and  $G = \langle \mathcal{S} \rangle$  is SF.

Output: a generating set for a completely reducible part of  $G$ .

- (1) Replace  $G$  by  $G^x$  in block upper triangular form with  $k$  diagonal blocks, where  $\mu(G_\rho^x)$  is abelian.
- (2)  $M := \text{NormalGenerators}(G_\rho)$ .
- (3) For  $i = 1$  to  $k$ , determine  $x_i$  such that  $\mu_i(G)^{x_i}$  is block upper triangular with completely reducible diagonal blocks, by the recursive calculation of fixed point spaces for  $\langle \mu_i(M) \rangle_u$ .
- (4) Return  $\mu(\mathcal{S}^y)$  where  $y = x \cdot \text{diag}(x_1, \dots, x_k)$ .

**Remark 4.1.** If  $G$  is nilpotent-by-finite then we can take  $k = 1$ ,  $\mu_1 = \text{id}$ , and omit step (1).

We need one other procedure for completely reducible  $G \leq \text{GL}(n, \mathbb{P})$ : `PresentationCR(G)` returns a presentation of  $G$ . This combines a presentation of  $\varphi_\rho(G)$ , computed using the machinery of [3], with `PresentationA(G_\rho)`.

#### 4.3. The unipotent radical

Our next procedure is based on Lemma 3.9 and its proof.

`RankOfUnipotentRadical(G)`

Input: a finite subset  $\mathcal{S} = \{g_1, \dots, g_r\}$  of  $\text{GL}(n, \mathbb{P})$  such that  $G = \langle \mathcal{S} \rangle$  is SF.

Output:  $\text{h}(U(G)) = \text{rk}(U(G))$ .

- (1)  $\tilde{G} := \langle \text{CompletelyReduciblePart}(G) \rangle$ .
- (2) Find  $X := \text{NormalGenerators}(U(G))$  from `PresentationCR`( $\tilde{G}$ ).
- (3) While  $\text{RankU}(\langle x, x^{g_i}, x^{g_i^{-1}} : x \in X, 1 \leq i \leq r \rangle) > \text{RankU}(\langle X \rangle)$  do  
 $X := \{x, x^{g_i}, x^{g_i^{-1}} : x \in X, 1 \leq i \leq r\}$ .
- (4) Return  $\text{RankU}(\langle X \rangle)$ .

**Remark 4.2.** The finitely generated subgroup  $H = \langle X \rangle$  of  $U(G)$  such that  $\text{rk}(H) = \text{rk}(U(G))$  found at the end of step (3) could be valuable in further computations with  $G$ .

#### 4.4. Algorithms for computing ranks, and an application

Guided by Remark 2.4, we assemble our constituent procedures into the final algorithms.

`HirschNumber(G)`

Input: a finite subset  $\mathcal{S}$  of  $\text{GL}(n, \mathbb{P})$  such that  $G = \langle \mathcal{S} \rangle$  is SF.

Output:  $\text{h}(G)$ .

Return  $\text{RankCR}(\langle \text{CompletelyReduciblePart}(G) \rangle) + \text{RankUnipotentRadical}(G)$ .

Then  $\text{RankBound}(G) := \text{HirschNumber}(G) + \text{rk}(\text{GL}(nm, 3))$  is an upper bound on the Prüfer rank of  $G$  (see Remark 2.10).

Corollary 3.3 gives us the following.

`IsOfFiniteIndex(G, H)`

Input: finite subsets  $\mathcal{S}_1, \mathcal{S}_2$  of  $\text{GL}(n, \mathbb{P})$  such that  $G = \langle \mathcal{S}_1 \rangle$  is SF and  $H = \langle \mathcal{S}_2 \rangle \leq G$ .

Output: true if  $|G : H|$  is finite; false otherwise.

Return true if  $\text{HirschNumber}(G) = \text{HirschNumber}(H)$ ; else return false.

#### 4.5. The implementation

We have implemented our algorithms as part of the MAGMA package INFINITE [9]. An algorithm of Biasse and Fieker [4] is used to work with irreducible abelian groups over number fields.

We report on several examples below (these will be available in a future release of INFINITE). Our experiments were performed on a 2 GHz machine using MAGMA V2.19-6. The test groups are conjugated to ensure that generators are not sparse and matrix entries are large. Each time has been averaged over three runs. As observed in [7,8], the single most expensive task is evaluating relators to obtain normal generators for the congruence subgroup.

- (1)  $G_1$  is an irreducible non-abelian subgroup of  $GL(2, \mathbb{Q}(i))$ ,  $i = \sqrt{-1}$ , and  $G_2 \leq GL(5, \mathbb{Q})$  is a solvable group from the database of maximal finite rational matrix groups [14]. Then  $G_3 = G_1 \otimes G_2$  is a 5-generator AF completely reducible subgroup of  $GL(10, \mathbb{Q}(i))$ . We compute  $h(G_3) = 3$  in 10s.
- (2)  $G_4 \leq G_3 \otimes UT(3, \mathbb{Z})$  is a 15-generator, nilpotent-by-finite (NF), reducible but not completely reducible subgroup of  $GL(30, \mathbb{Q}(i))$ . We compute  $h(G_4) = 6$  in 87s.
- (3)  $G_5 \leq H \otimes T$  where  $T$  is an upper triangular subgroup of  $GL(6, \mathbb{Q})$  and  $H = \text{diag}(H_1, H_2)$ ;  $H_1, H_2$  are maximal finite rational matrix groups of degrees 4, 2 respectively. The 8-generator group  $G_5$  is SF and not NF. We compute  $h(G_5) = 7$  in 1104s, and establish that a random 4-generator subgroup has infinite index in 163s.
- (4) Let  $a \in GL(6, \mathbb{Q})$  be of the form  $\text{diag}(1, 2, \dots)$  and let  $b = \begin{pmatrix} x & y \\ 0 & u \end{pmatrix}$  where  $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $y$  is a non-zero  $2 \times 4$  matrix over  $\mathbb{Q}$ , and  $u \in UT(4, \mathbb{Z})$ . Then  $G_6 \leq GL(6, \mathbb{Q}(\sqrt{5}))$  is conjugate to a group generated by  $a, b$ , another diagonal matrix and two other unipotent matrices in  $GL(6, \mathbb{Q})$ . Note that  $G_6$  is SF but not PF. We compute  $h(G_6) = 12$  in 18s.
- (5) For each of  $G_3, G_4, G_6$  we select random finitely generated non-cyclic subgroups  $\hat{G}_j$ . To establish that  $\hat{G}_j$  has finite index in  $G_j$  takes 4s, 53s, and 17s respectively.

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