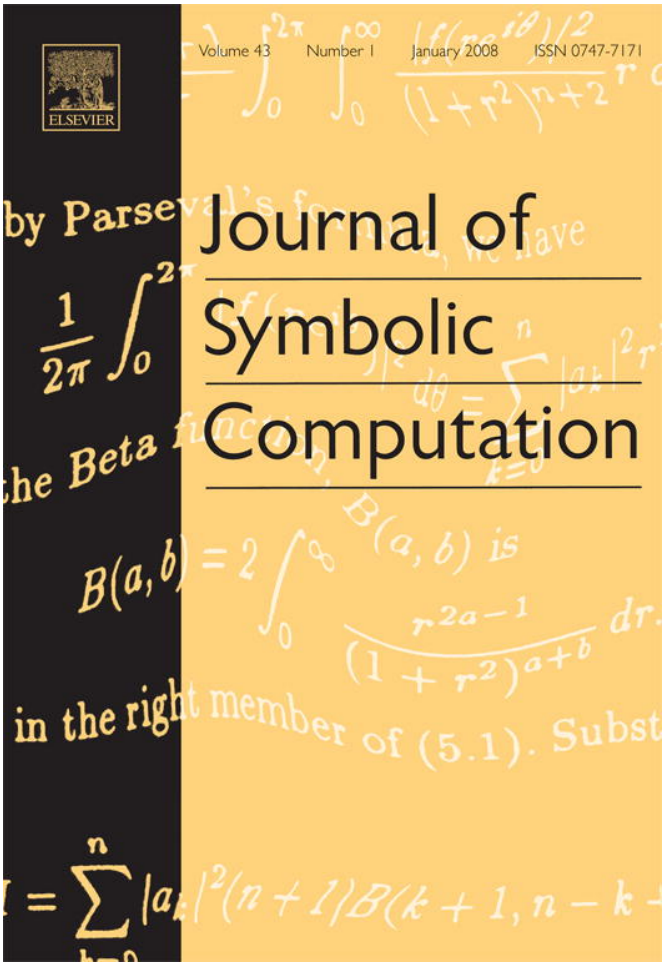

Available online at www.sciencedirect.com
ScienceDirect
Journal of Symbolic Computation 43 (2008) 8–26
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Algorithms for computing with nilpotent matrix groups over infinite domains

A.S. Detinko, D.L. Flannery [1]

Department of Mathematics, National University of Ireland, Galway, Ireland

Received 27 February 2007; accepted 7 August 2007
Available online 17 August 2007

Abstract

We develop methods for computing with matrix groups defined over a range of infinite domains, and apply those methods to the design of algorithms for nilpotent groups. In particular, we provide a practical nilpotency testing algorithm for matrix groups over an infinite field. We also provide algorithms to answer a number of structural questions for a nilpotent matrix group.The main algorithms have been implemented in GAP, for groups over the rational number field.

Keywords: Matrix group; Nilpotency testing; Nilpotent group; Infinite field

1. Introduction

In this paper we develop a technique for computing with matrix groups defined over infinite domains, based on changing the ground domain via congruence homomorphism. This technique has proved to be an efficient tool in linear group theory (see e.g. Dixon (1971, Chapter 3)). It is particularly useful for handling finitely generated linear groups (see Wehrfritz (1973, Chapter 4)), and affords a general approach to many computational problems for infinite matrix groups. We apply the technique here to algorithms for nilpotent matrix groups.

Let $\mathbb{F}$ be a field and let $G \leq \mathrm{GL}(n, \mathbb{F})$ be given by a finite generating set. We obtain algorithms for carrying out the following tasks:

(i) testing nilpotency of G;

E-mail addresses: alla.detinko@nuigalway.ie (A.S. Detinko), dane.flannery@nuigalway.ie (D.L. Flannery).
[1] Tel.: +353 91 493587; fax: +353 91 494542.

0747-7171/$ - see front matter
doi:10.1016/j.jsc.2007.08.001

and, if G is nilpotent,

(ii) constructing a polycyclic presentation of G;
(iii) testing whether G is completely reducible, and finding a completely reducible series of G-modules;
(iv) deciding finiteness of G, and calculating $|G|$ if G is finite;
(v) finding the p-primary decomposition of G, and finding all Sylow p-subgroups if G is finite.

These algorithms address standard problems in computational group theory (i, ii, iv) and computing with matrix groups (iii), and facilitate structural investigation of nilpotent linear groups (v).

Our guiding objective has been to design algorithms that cover the broadest possible range of infinite domains. However, for convenience or reasons of practicality we sometimes place restrictions on $\mathbb{F}$. For example, a preliminary reduction in nilpotency testing assumes that $\mathbb{F}$ is perfect; and constructing polycyclic presentations requires $\mathbb{F}$ to be finite or an algebraic number field. Two important types of field that we treat throughout the paper are algebraic number fields and certain functional fields. Implementation and practicality of our algorithms depend on the machinery (such as polynomial factorization) that is available for computing with the various fields.

A finitely generated subgroup of $\mathrm{GL}(n, \mathbb{F})$ is contained in $\mathrm{GL}(n, R)$ for some finitely generated integral domain $R \subseteq \mathbb{F}$. In turn, each completely reducible solvable subgroup of $\mathrm{GL}(n, R)$ is finitely generated (by Dixon (1971, Theorem 6.4, p. 111) and Wehrfritz (1973, 4.10, p. 57)). Our algorithms therefore return information not only about finitely generated nilpotent subgroups of $\mathrm{GL}(n, \mathbb{F})$, but also about nilpotent subgroups of $\mathrm{GL}(n, R)$ for any finitely generated integral domain R. Furthermore, these algorithms may be regarded as a platform for computing in abstract finitely generated nilpotent groups, because such a group is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$. A method for constructing a representation of a finitely presented polycyclic group in $\mathrm{GL}(n, \mathbb{Z})$ can be found in Lo and Ostheimer (1999).

Nilpotency is an important group-theoretic property, and testing nilpotency is consequently one of the basic functions of any computational group theory system. This paper provides the first uniform and effective solution to the problem of computing with infinite nilpotent matrix groups. Our algorithms for nilpotency testing (over finite fields, and $\mathbb{Q}$) have been implemented as part of the GAP package 'Nilmat' (Detinko et al., 2007) (this is joint work with Bettina Eick). Standard algorithms for nilpotency testing in GAP and MAGMA sometimes fail to decide nilpotency even for small finite matrix groups, and fail for almost all infinite matrix groups. In the paper's final subsection we give some experimental results, and other details of the 'Nilmat' package.

2. Related results

Computing in matrix groups over infinite domains is a relatively new area of computational group theory. Most of the algorithms in this area are concerned with classes of solvable-by-finite groups (see Assmann and Eick (2005, 2007), Beals (1999) and Ostheimer (1999)). Solvable-by-finite groups constitute the more optimistic class of the Tits alternative. The other class consists of groups containing a non-abelian free subgroup. For those groups, some basic computational problems, such as membership testing and construction of presentations, are undecidable (see Beals (1999), Dixon (1985) and Eick (2005)).

Changing the ground domain is a standard technique in linear group theory. In certain specialized situations it has been used by several authors for computing with matrix groups;

see e.g. Luks (1992). In Beals (1999), a generalization of the technique as in Luks (1992) leads to a Monte Carlo solvability testing algorithm for potentially infinite subgroups G of $\mathrm{GL}(n, \mathbb{Q})$. The algorithm accepts as input a finite set S of generators of G, and tests solvability of $\psi_p(G) \leq \mathrm{GL}(n, p)$, where ψ_p is reduction modulo a prime p not dividing the denominators of the entries of the elements in $S \cup S^{-1}$. It is shown in Beals (1999) that there are only finitely many primes p such that $\psi_p(G)$ is solvable while G is not; so a non-solvable group will be identified as solvable by the algorithm of Beals (1999) with small probability.

The ideas of Beals (1999) can be applied to nilpotency testing. However, in contrast to solvability testing, nilpotency testing by these means is not as reliable: the upper bound on nilpotency class for nilpotent subgroups of $\mathrm{GL}(n, q)$ can be much larger than the bound for nilpotent subgroups of $\mathrm{GL}(n, \mathbb{Q})$ (see Bialostocki (1986) and Wehrfritz (2001)). Hence the solvability testing arguments of Beals (1999) may not be efficient when applied to nilpotency testing, if one simply replaces bounds depending on derived length by bounds depending on nilpotency class.

To obtain a deterministic algorithm for testing solvability of a finitely generated matrix group G over $\mathbb{Q}$, it is necessary to test solvability both of the kernel G_p and of the image $\psi_p(G)$ of a reduction mod p homomorphism ψ_p. Theoretical background for doing this is laid out in Dixon (1985), where G_p is described for solvable-by-finite $G \leq \mathrm{GL}(n, \mathbb{Q})$. Using those results, a deterministic algorithm for solvability testing was proposed in Ostheimer (1999). There were two main obstacles to a full implementation of the algorithm in Ostheimer (1999): solvability testing of matrix groups over a finite field; and efficient construction of G_p. Practical solutions of both of these problems were obtained in Assmann and Eick (2005) and Assmann (2003). Specifically, Assmann and Eick (2005) contains a method to construct a polycyclic presentation of $\psi_p(G)$, and thereby to test solvability of $\psi_p(G)$. The relators of this presentation may be used to calculate generators of a subgroup of G whose normal closure is G_p. Although the algorithm has bottlenecks (see Assmann and Eick (2005, p. 1281)), it has been successfully implemented for solvability testing over finite and algebraic number fields (see the 'Polenta' package (Assmann and Eick, 2007) in GAP (The GAP group, 2006)).

The main aims of Assmann and Eick (2005) are to test whether a finitely generated subgroup G of $\mathrm{GL}(n, \mathbb{Q})$ is polycyclic, and, if so, to construct a polycyclic presentation for G. Complete solutions of those problems are given in a subsequent publication (Assmann and Eick, 2007). This provides an avenue for testing nilpotency of G: if G is not polycyclic then it is not nilpotent; otherwise, nilpotency of G can be tested using a polycyclic presentation of G (for which see e.g. Lo (1998, Section 4)).

In this paper we propose an essentially different approach to nilpotency testing, applicable over a broad range of infinite domains. In contrast to Assmann and Eick (2007), our algorithms do not require *a priori* testing of polycyclicity and computation of polycyclic presentations, and are designed directly for nilpotency testing.

We rely on methods and results of linear group theory, especially structural results for nilpotent linear groups (Suprunenko (1976, Chapter VII), Detinko and Flannery (2005, 2006b)). Accordingly, a feature of our algorithms is that they return detailed structural information about input nilpotent groups. A full solution of the problem of testing nilpotency over finite fields appears in Detinko and Flannery (2006a) (as we will see, much of Detinko and Flannery (2006a) remains valid over any field). To transfer nilpotency testing to the case of groups over a finite field, we use a congruence homomorphism with torsion-free kernel; see Section 3. Other methods that transfer nilpotency testing to the case of finite groups are given in Section 4.5.

3. Changing the ground domain via congruence homomorphism

In this section we present some results from linear group theory that form the theoretical background for our algorithms.

First we set up some notation. Let Δ be an integral domain. For any ideal ϱ of Δ, the natural surjection $\psi_\varrho : \Delta \to \Delta/\varrho$ extends entrywise to a matrix ring homomorphism $\mathrm{Mat}(n, \Delta) \to \mathrm{Mat}(n, \Delta/\varrho)$, and then restricts to a group homomorphism $\mathrm{GL}(n, \Delta) \to \mathrm{GL}(n, \Delta/\varrho)$, which we also denote ψ_ϱ. The map ψ_ϱ on $\mathrm{GL}(n, \Delta)$ is called a Minkowski or congruence homomorphism (Suprunenko (1976, p. 65)), and its kernel is called a (principal) congruence subgroup of $\mathrm{GL}(n, \Delta)$. We denote the congruence subgroup corresponding to ϱ by $\mathcal{G}_\varrho$, or $\mathcal{G}(n, \Delta, \varrho)$ in more detail. If $G \leq \mathrm{GL}(n, \Delta)$ then $G_\varrho := G \cap \mathcal{G}_\varrho$. For an integer m we write $m \in \varrho$ to mean that $m \cdot 1_\Delta \in \varrho$.

We are interested in domains Δ and ideals $\varrho \subseteq \Delta$ such that $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free if $\mathrm{char}\,\Delta = 0$, or each torsion element of $\mathcal{G}(n, \Delta, \varrho)$ is unipotent if $\mathrm{char}\,\Delta > 0$. Such domains in characteristic zero are discussed in Suprunenko (1976, Chapter III, Section 11). A slight modification of the proofs in Suprunenko (1976) takes care of the positive characteristic case. To keep the account here reasonably self-contained, we give full proofs of both cases.

Lemma 3.1. *Let Δ be a unique factorization domain, $q \in \Delta$ be irreducible, and ϱ be the principal ideal $q\Delta$ of Δ. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ has non-trivial torsion elements. Then*

(i) *there is a unique prime $p \in \mathbb{Z}$ such that $p \in \varrho$;*
(ii) *for some $b \in \mathrm{Mat}(n, \Delta)$, $pb = -\sum_{i=2}^{p} \binom{p}{i} q^{i-1} b^i$; and*
(iii) *every torsion element of $\mathcal{G}(n, \Delta, \varrho)$ has p-power order.*

Proof (*Cf. Suprunenko (1976), Proof of Theorem 3, pp. 68–69*). Let $h \in \mathcal{G}_\varrho$ be of prime order p. We have $h = 1_n + qb$ for some $b \in \mathrm{Mat}(n, \Delta)$. Then

$$1_n = h^p = 1_n + pqb + \cdots + \binom{p}{i} q^i b^i + \cdots + q^p b^p$$

where the binomial coefficients are read modulo $\mathrm{char}\,\Delta$. Hence

$$pb = -\sum_{i=2}^{p} \binom{p}{i} q^{i-1} b^i \tag{1}$$

and it follows that either q divides p, or q divides every entry of b.

Suppose that q does not divide p. Then for some integer $\alpha \geq 1$, q^α divides every entry of b, but $q^{\alpha+1}$ does not. Then (1) implies that $q^{2\alpha+1}$ divides pb, a contradiction. Thus q divides p.

If $\mathcal{G}_\varrho$ contains a non-trivial element of p'-order then it contains an element of prime order $r \neq p$. By the preceding one, then, q divides both p and r and hence divides $1 = px + ry$ for some $x, y \in \mathbb{Z}$ guaranteed by Bézout's lemma. But q is not a unit by definition. Thus every torsion element of $\mathcal{G}_\varrho$ is a p-element. □

Proposition 3.2. *Let Δ, q, and ϱ be as in Lemma 3.1.*

(i) *If $\mathrm{char}\,\Delta = t > 0$ then every torsion element of $\mathcal{G}(n, \Delta, \varrho)$ is a t-element.*
(ii) *Suppose that $\mathrm{char}\,\Delta = 0$, q does not divide 2, and q^2 does not divide p for any prime $p \in \mathbb{Z}$. Then $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free.*

Proof (*Cf. Suprunenko (1976), pp. 68–69*). (i) Follows from parts (i) and (iii) of Lemma 3.1.

(ii) Supposing that $\mathcal{G}_\varrho$ has non-trivial torsion, then $p = qr$ for some odd prime p and $r \in \Delta$ not divisible by q. By Lemma 3.1(ii), for some $b, c \in \mathrm{Mat}(n, \Delta)$ we have $qrb = q^2b^2c$. Hence q^α divides every entry of b for some $\alpha \geq 1$ such that $q^{\alpha+1}$ does not divide every entry of b. As q does not divide r, $rb = qb^2c$ yields the contradiction that $q^{2\alpha+1}$ divides every entry of b. □

The next result mimics Lemma 3.1.

Lemma 3.3. *Let Δ be a Dedekind domain, and let ϱ be a proper prime ideal (that is, maximal ideal) of Δ. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ has non-trivial torsion elements. Then*

(i) *there is a unique prime $p \in \mathbb{Z}$ such that $p \in \varrho$;*
(ii) *for some $b \in \mathrm{Mat}(n, \varrho)$, $pb_{j,k} = -\sum_{i=2}^{p} \binom{p}{i} b_{j,k}^{(i)}$, where $b_{j,k}^{(i)}$ denotes the (j,k)th entry of b^i; and*
(iii) *every torsion element of $\mathcal{G}(n, \Delta, \varrho)$ has p-power order.*

Proof (*Cf. Suprunenko (1976), Proof of Theorem 4, p. 70*). If $h \in \mathcal{G}_\varrho$ is a torsion element of prime order p then

$$pb + \cdots + \binom{p}{i} b^i + \cdots + b^p = 0_n$$

for some $b \in \mathrm{Mat}(n, \varrho)$, reading the binomial coefficients modulo char Δ. Now (ii) is clear.

Let $l \geq 1$ be the integer such that $b_{j,k} \in \varrho^l$ for all j, k, but $b_{r,s} \notin \varrho^{l+1}$ for some r, s. (Note that such an integer l definitely exists, because the ideal I of Δ generated by the entries of b is contained in ϱ, $I = \varrho J$ where J is the ideal $\varrho^{-1}I$, and J has a maximal power of ϱ in its primary decomposition.) Then (ii) and $b_{j,k}^{(i)} \in \varrho^{il}$ imply that $pb_{j,k} \in \varrho^{2l}$. Suppose that $p \notin \varrho$. Since ϱ is a maximal ideal of Δ, we have that Δ is generated by p and ϱ. Let $x \in \Delta$, $y \in \varrho$ be such that $px + y = 1$. Then $b_{j,k} = pb_{j,k}x + b_{j,k}y \in \varrho^{2l} + \varrho^l\varrho \subseteq \varrho^{l+1}$, a contradiction. Thus $p \in \varrho$. Moreover, p is the unique prime integer such that $p \in \varrho$ (otherwise $1 \in \varrho$ by Bézout), so that every torsion element of $\mathcal{G}_\varrho$ is a p-element. □

Proposition 3.4. *Let Δ and ϱ be as in Lemma 3.3.*

(i) *If* char $\Delta = t > 0$ *then every torsion element of $\mathcal{G}(n, \Delta, \varrho)$ is a t-element.*
(ii) *If* char $\Delta = 0$, $2 \notin \varrho$ *and $p \notin \varrho^2$ for all primes $p \in \mathbb{Z}$, then $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free.*

Proof (*Cf. Suprunenko (1976), p. 70*). (i) This follows at once from Lemma 3.3(i) and (iii).

(ii) If $\mathcal{G}_\varrho$ has non-trivial torsion then $\mathcal{G}_\varrho$ has elements of p-power order, where $p \in \varrho$ for an odd prime p. By Lemma 3.3(ii), there exist an element b of Δ and an integer l such that $b \in \varrho^l \setminus \varrho^{l+1}$ (so that ϱ^l is the largest power of ϱ appearing in the primary decomposition of the ideal $b\Delta$) and $pb \in \varrho^{2l+1}$. Certainly then $pb \in \varrho^{l+2}$.

We now establish a contradiction. First, $pb \in \varrho^{l+2}$ implies that $p\Delta \cdot b\Delta \subseteq pb\Delta \subseteq \varrho^{l+2}$, so ϱ^{l+2} appears in the primary decomposition of $p\Delta \cdot b\Delta$. But since $p \in \varrho \setminus \varrho^2$, we know that ϱ^{l+1} is the largest power of ϱ appearing in this decomposition. Hence $pb \notin \varrho^{l+2}$. We conclude that $\mathcal{G}_\varrho$ is torsion-free in this case. □

To round out this section, we look briefly at how congruence homomorphisms may be applied in practice to finitely generated matrix groups. The image of the homomorphism should be a matrix group for which solutions to the specific problems are known (for example, the image is

over a finite field), and the kernel of the homomorphism should be either torsion-free or consist of unipotent elements.

Let $\mathbb{F}$ be the field of fractions of the integral domain Δ, and let R be a finitely generated subring of $\mathbb{F}$. In particular, if $G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, \mathbb{F})$ then $R = R(G)$ denotes the ring generated by the entries of the elements of $\{g_i, g_i^{-1} \mid 1 \leq i \leq r\}$. Obviously $G \leq \mathrm{GL}(n, R(G))$.

Fix a finite generating set of R, and let $\pi \subseteq \Delta$ be the set of denominators of the generators. Denote by Δ_π the ring of fractions with denominators in the submonoid of $\Delta^\times$ generated by π (Cohn, 1989, p. 311). Of course, $R \subseteq \Delta_\pi$. If Δ is a UFD or Dedekind domain then Δ_π is a UFD or Dedekind domain, respectively (Cohn (1989, Theorem 3.7, p. 315 and Corollary 5.2, p. 322)). Since the quotient of a finitely generated commutative ring by a maximal ideal is a finite field (Wehrfritz (1973, 4.1, p. 50)), if Δ is finitely generated and ϱ is a maximal ideal of Δ_π then Δ_π/ϱ is a finite field. Thus, if $G \leq \mathrm{GL}(n, \mathbb{F})$ then $\psi_\varrho : \mathrm{GL}(n, \Delta_\pi) \to \mathrm{GL}(n, \Delta_\pi/\varrho)$ maps G into some $\mathrm{GL}(n, q)$.

Now we look at two examples that are of specific interest from a computational point of view.

Example 3.5. Let $\mathbb{F}$ be an algebraic number field, and let Δ be the ring of integers of $\mathbb{F}$. Since Δ is finitely generated, Δ_π is a finitely generated Dedekind domain. Let ϱ be a maximal (i.e. proper prime) ideal of Δ_π not containing 2, such that $p \notin \varrho^2$ for all primes $p \in \mathbb{Z}$; then $\mathcal{G}(n, \Delta_\pi, \varrho)$ is torsion-free by Proposition 3.4, and Δ_π/ϱ is a finite field. In particular, if $\mathbb{F} = \mathbb{Q}$ then $\Delta = \mathbb{Z}$, and if we choose an odd prime $p \in \mathbb{Z}$ which does not divide any element of π then $\varrho = p\Delta_\pi$ is as required. In this case $\Delta_\pi/\varrho = \mathrm{GF}(p)$.

For number fields $\mathbb{F}$ in general, to find ϱ we can reduce to $\mathbb{Q}$ after selecting a $\mathbb{Q}$-basis of $\mathbb{F}$; this however has the disadvantage of blowing up the size of matrices. An alternative method is as follows. Suppose that $\mathbb{F} = \mathbb{Q}(\alpha)$ contains all generators of R, where α is an algebraic integer. Let m be the degree of the minimal polynomial of α. Expressing each generator of R uniquely as a $\mathbb{Q}$-linear combination of $\{1, \alpha, \ldots, \alpha^{m-1}\}$, and thereafter obtaining each generator in the form β/z where β is an algebraic integer and $z \in \mathbb{Z}$, we can easily find $\pi \subseteq \mathbb{Z}$. If $p \in \mathbb{Z}$ is an odd prime element of Δ not dividing any element of π then $\varrho = p\Delta_\pi$ is a maximal ideal of Δ_π such that $\Delta_\pi/\varrho = \mathrm{GF}(p^l)$ for some $l \leq m$, and $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free. Note that $R \leq \mathbb{Z}_\pi[\alpha]$. So the reduction mod p congruence homomorphism on a finitely generated subgroup G of $\mathrm{GL}(n, \mathbb{F})$ with $R = R(G)$ is easy to describe. That is, to evaluate ψ_ϱ, we reduce elements of $\mathbb{Z}_\pi$ mod p, and if $f(X) \in \mathbb{Z}[X]$ is the minimal polynomial of α then $\psi_\varrho(\alpha)$ is a root of the mod p-reduction $\bar{f}(X)$ of $f(X)$. If $\bar{f}(X)$ is irreducible over $\mathrm{GF}(p)$ then $l = m$; otherwise $l < m$.

Example 3.6. Let $\mathbb{F}$ be a function field $P(X)$, and let Δ be the polynomial ring $P[X]$, where P is a UFD. Then Δ is a UFD (Cohn (1989, p. 316)), and therefore so too is Δ_π. Let $q = X - \alpha$, where α is not a root of any element of π. (If P is infinite then of course α always exists in P; else we can replace the finite field P by a finite extension containing α.) Then $\varrho = q\Delta_\pi$ is a prime ideal of Δ_π. By Proposition 3.2, either $\mathcal{G}(n, \Delta_\pi, \varrho)$ is torsion-free, or every torsion element of $\mathcal{G}(n, \Delta_\pi, \varrho)$ is unipotent. The effect of ψ_ϱ is just substitution of α for the indeterminate X in elements of Δ_π. Hence $\psi_\varrho(\Delta_\pi)$ can be regarded as a subring of P. If P is finite then $\psi_\varrho(\Delta_\pi)$ is also a finite field. When P has characteristic zero we can apply a suitable congruence homomorphism over the finitely generated integral domain $\psi_\varrho(\Delta_\pi) \subseteq P$, in line with the following simple observation: if $\psi_{\varrho_1} : \Delta \to \Delta/\varrho_1$ and $\psi_{\varrho_2/\varrho_1} : \Delta/\varrho_1 \to \Delta/\varrho_2$ are (natural) homomorphisms of integral domains such that $\mathcal{G}(n, \Delta, \varrho_1)$ and $\mathcal{G}(n, \Delta/\varrho_1, \varrho_2/\varrho_1)$ are both torsion-free, then $\mathcal{G}(n, \Delta, \varrho_2)$ is torsion-free. As an example, take $P = \mathbb{Q}$. Here $\psi_\varrho(\Delta_\pi) \subseteq \mathbb{Z}_{\pi_1}$ for some finite subset π_1 of $\mathbb{Z} \setminus \{0\}$, and we are back to the situation of Example 3.5.

4. Computing with nilpotent matrix groups

In this section we proceed to the design of algorithms for computing with matrix groups over a field $\mathbb{F}$, as set out in the introduction. We are guided to a large extent by the algorithms and results in Detinko and Flannery (2006a). Although only finite fields $\mathbb{F}$ were treated in Detinko and Flannery (2006a), most of that paper's fundamental results are valid over any field $\mathbb{F}$.

4.1. Splitting nilpotent linear groups

In linear group theory it is common practice first to reduce problems to the completely reducible case. This reduction is more straightforward for nilpotent linear groups than it is for arbitrary linear groups (see e.g. Detinko and Flannery (2005, Subsection 2.1)). In this subsection we consider a computational approach to the reduction.

Our starting point is the Jordan decomposition. Recall that $h \in \mathrm{GL}(n, \mathbb{F})$ is said to be diagonalizable if h is conjugate to a diagonal matrix over some extension field of $\mathbb{F}$, and h is semisimple if $\langle h \rangle \leq \mathrm{GL}(n, \mathbb{F})$ is completely reducible. A semisimple element of $\mathrm{GL}(n, \mathbb{F})$ need not be diagonalizable, unless $\mathbb{F}$ is perfect: then the two concepts coincide. Denote the algebraic closure of $\mathbb{F}$ by $\overline{\mathbb{F}}$. For each $g \in \mathrm{GL}(n, \mathbb{F})$, there is a unique unipotent matrix $g_u \in \mathrm{GL}(n, \overline{\mathbb{F}})$ and a unique diagonalizable matrix $g_s \in \mathrm{GL}(n, \overline{\mathbb{F}})$ such that $g = g_s g_u = g_u g_s$ (see Wehrfritz (1973, 7.2, p. 91)). Note that this Jordan decomposition of g depends only on g i.e. it is the same over each and every extension of $\mathbb{F}$. If $\mathbb{F}$ is perfect then by Segal (1983, Proposition 1, p. 134), g_u and g_s are both in $\mathrm{GL}(n, \mathbb{F})$.

An algorithm to compute the Jordan decomposition can be found in Babai et al. (1996, Appendix A). Systems such as GAP also contain standard functions for computing the decomposition.

Let G be a finitely generated subgroup of $\mathrm{GL}(n, \mathbb{F})$, say $G = \langle g_1, \ldots, g_r \rangle$. Define

$$G_u = \langle (g_1)_u, \ldots, (g_r)_u \rangle \quad \text{and} \quad G_s = \langle (g_1)_s, \ldots, (g_r)_s \rangle.$$

Since $g_i = (g_i)_u (g_i)_s \in \langle G_u, G_s \rangle$, clearly $G \leq G^* := \langle G_u, G_s \rangle$. In general, neither G_u nor G_s is necessarily a subgroup of G.

Lemma 4.1. (i) *G is nilpotent if and only if G_u, G_s are nilpotent and $[G_u, G_s] = 1$.*
(ii) *If G is nilpotent then $G \leq G^* = G_u \times G_s$.*

Proof. If G is nilpotent then the assignments $g \mapsto g_u$ and $g \mapsto g_s$ define homomorphisms $G \to G_u$ and $G \to G_s$; furthermore $G^* = G_u \times G_s$ (see Segal (1983, Proposition 3, p. 136)). On the other hand, if G_u, G_s are nilpotent and $[G_u, G_s] = 1$, then G^* and thus $G \leq G^*$ are nilpotent. □

Remark 4.2. Let G be nilpotent. Then $G_u = \{g_u \mid g \in G\}$ and $G_s = \{g_s \mid g \in G\}$. Also, sometimes $G = G_u \times G_s$. For example, this is true if $\mathbb{F}$ is finite. As another example, if G is an algebraic group (over algebraically closed $\mathbb{F}$) then $g_u, g_s \in G$ for all $g \in G$, so that $G = G^*$.

Lemma 4.3. *G_u is nilpotent if and only if it is unipotent, that is, conjugate to a subgroup of the group $\mathrm{UT}(n, \mathbb{F})$ of all upper unitriangular matrices over $\mathbb{F}$.*

Proof. A unipotent group is unitriagularizable (see Wehrfritz (1973, 1.21, p. 14)). If G_u is nilpotent then $G_u = \{g_u \mid g \in G\}$ is unipotent. As is well-known, $\mathrm{UT}(n, \mathbb{F})$ is nilpotent (of class $n - 1$). □

In Detinko and Flannery (2006a, Subsection 2.1), a recursive procedure is given for deciding whether a group generated by unipotent matrices (over any field $\mathbb{F}$) is unipotent. We label that procedure `IsUnipotent` here.

`IsUnipotent`(H)

Input: $H = \langle h_1, \ldots, h_r \rangle$, $h_i \in \mathrm{GL}(n, \mathbb{F})$ unipotent, $\mathbb{F}$ any field.
Output: a $\mathrm{UT}(n, \mathbb{F})$-representation of H, or a message 'false' meaning that H is not unipotent.

Lemmas 4.1 and 4.3, and `IsUnipotent`, equate nilpotency testing of $G \leq \mathrm{GL}(n, \mathbb{F})$ to testing nilpotency of G_s and testing whether $[G_u, G_s] = 1$.

If G_u is unipotent then `IsUnipotent` finds a $\mathrm{UT}(n, \mathbb{F})$-representation of G_u by constructing a series

$$V = V_0 > V_1 > \cdots > V_{l-1} > V_l = 0 \tag{2}$$

of G_u-submodules of the underlying space V for $\mathrm{GL}(n, \mathbb{F})$, such that G_u acts trivially on each factor V_{i-1}/V_i. In fact, V_{i-1}/V_i is the fixed point space $\mathrm{Fix}_{G_u}(V/V_i)$. We get more when G is nilpotent, by the next two lemmas.

Lemma 4.4. *Each unipotent element of a completely reducible nilpotent subgroup of* $\mathrm{GL}(n, \mathbb{F})$ *is trivial.*

Proof. This follows from Suprunenko (1976, Corollary 1, p. 239). □

Lemma 4.5. *Let* $G \leq \mathrm{GL}(n, \mathbb{F})$ *be nilpotent,* $\mathbb{F}$ *a perfect field. Then*

(i) G_s *is completely reducible over* $\mathbb{F}$*;*
(ii) G *is completely reducible over* $\mathbb{F}$ *if and only if* $G_u = 1$.

Proof. A solvable matrix group of diagonalizable matrices (over any field) is completely reducible by Suprunenko (1976, Theorem 5, p. 172). Since $G_s \leq \mathrm{GL}(n, \mathbb{F})$ consists entirely of diagonalizable matrices, if $G_u = 1$ then $G = G_s$ is completely reducible. The converse is Lemma 4.4. □

If G is nilpotent and $\mathbb{F}$ is perfect then Lemmas 4.1 and 4.5 imply that each factor V_{i-1}/V_i of the series (2) is a completely reducible G^*-module. Now a subgroup of a nilpotent completely reducible subgroup of $\mathrm{GL}(n, \mathbb{F})$ is completely reducible by Suprunenko (1976, Theorem 5, p. 239), so we see that if G is nilpotent then `IsUnipotent` constructs completely reducible modules not just for G^* but also for G.

We now give a procedure for reducing nilpotency testing of $G \leq \mathrm{GL}(n, \mathbb{F})$ to testing nilpotency of a matrix group generated by diagonalizable matrices.

`Reduction`(G)

Input: $G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, \mathbb{F})$, $\mathbb{F}$ any field.
Output: G_s, a $\mathrm{UT}(n, \mathbb{F})$-representation of G_u, and a message that $[G_u, G_s] = 1$; or a message 'false' meaning that G is not nilpotent.

for $i \in [1 \,..\, r]$ do
  find $(g_i)_u$, $(g_i)_s$;
$G_u := \langle (g_i)_u : 1 \leq i \leq r \rangle$, $G_s := \langle (g_i)_s : 1 \leq i \leq r \rangle$;

if `IsUnipotent`(G_u) = 'false'
  then return 'false';
else $N := [G_u, G_s]$;
if $N \neq 1$
  then return 'false';
else return G_s.

There are other reductions to the completely reducible case. For example, we may proceed by first computing the radical R of the enveloping algebra $\langle G \rangle_{\mathbb{F}}$, and then the radical series

$$V \supset RV \supset R^2V \supset \cdots \supset R^mV = 0$$

(for methods to compute R, see e.g. Rónyai (1993)). Each term R^iV in this series is a G-module, and each factor $R^iV/R^{i+1}V$ is a completely reducible G-module. We can use the radical series to write G in block upper triangular form, thereby obtaining a homomorphism θ of G onto a completely reducible subgroup of $\mathrm{GL}(n, \mathbb{F})$. If G is nilpotent then $\ker\theta$ is the unipotent radical of G (the unique maximal unipotent normal subgroup of G), and $\ker\theta$ commutes with every diagonalizable element of G (see Wehrfritz (1973, 7.11, p. 97)).

4.2. *Nilpotency testing of matrix groups via change of the ground domain: First steps*

We now prepare the way for applying Section 3 to nilpotency testing over an arbitrary field $\mathbb{F}$.

Denote by $Z_i(G)$ the ith term of the upper central series of G. That is, $Z_0(G) = 1$, and $Z_i(G)/Z_{i-1}(G) = \mathsf{Z}(G/Z_{i-1}(G))$.

Lemma 4.6. *If G is a completely reducible nilpotent subgroup of* $\mathrm{GL}(n, \mathbb{F})$ *then* $|G : \mathsf{Z}(G)|$ *is finite.*

Proof. See Dixon (1971, Corollary 6.5, p. 114), or Suprunenko (1976, Theorem 1, p. 208). Also cf. Zassenhaus' result (Wehrfritz, 1973, 3.4, p. 44). □

Remark 4.7. Suprunenko (1976, Theorem 1, p. 208) is stated for irreducible groups only. The result for completely reducible nilpotent subgroups G of $\mathrm{GL}(n, \mathbb{F})$ follows from this, because $G/\mathsf{Z}(G)$ is isomorphic to a subgroup of the direct product of central quotients of irreducible nilpotent linear groups (each of degree no more than n).

Lemma 4.8. *Let G be a completely reducible nilpotent subgroup of* $\mathrm{GL}(n, \mathbb{F})$. *If N is a torsion-free normal subgroup of G then $N \leq \mathsf{Z}(G)$.*

Proof. Suppose that $N \not\leq \mathsf{Z}(G)$. Then $N\mathsf{Z}(G)/\mathsf{Z}(G)$ is a non-trivial normal subgroup of the nilpotent group $G/\mathsf{Z}(G)$, so it has non-trivial intersection with $Z_2(G)/\mathsf{Z}(G)$. Let $x \in N \cap Z_2(G)$, $x \notin \mathsf{Z}(G)$. By Lemma 4.6, $x^m \in \mathsf{Z}(G)$ for some m. Select $g \in G$ such that $x^g = x\varepsilon$ for some $\varepsilon \in \mathsf{Z}(G)$, $\varepsilon \neq 1$. Then $x^m = (x^m)^g = (x^g)^m = x^m\varepsilon^m$ implies that ε is a non-trivial torsion element of G. But $\varepsilon = x^{-1}x^g \in N$. Hence N must indeed be contained in $\mathsf{Z}(G)$. □

Now let G be a finitely generated subgroup of $\mathrm{GL}(n, \mathbb{F})$. Suppose that Δ is a finitely generated subring of $\mathbb{F}$ such that $G \leq \mathrm{GL}(n, \Delta)$, and let ϱ be an ideal of Δ. We continue with the notation $\mathcal{G}(n, \Delta, \varrho)$ and G_ϱ adopted in Section 3 for congruence subgroups. Without loss of generality, we may assume that $\mathbb{F}$ is the field of fractions of Δ.

Lemma 4.9. *Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free if char $\Delta = 0$, and all torsion elements of $\mathcal{G}(n, \Delta, \varrho)$ are unipotent if char $\Delta > 0$. Let G be completely reducible as a subgroup of* $\mathrm{GL}(n, \mathbb{F})$. *If G is nilpotent then G_ϱ is a torsion-free central subgroup of G.*

Proof. By Lemma 4.4 and the hypotheses, G_ϱ is torsion-free. Then the result follows from Lemma 4.8. □

In the discussion at the end of Section 3, we gave examples of selecting ϱ as in Lemma 4.9 for various $\mathbb{F}$ and Δ. Also, Section 4.1 shows how to split off a completely reducible subgroup of $\mathrm{GL}(n, \mathbb{F})$ from an arbitrary finitely generated nilpotent subgroup of $\mathrm{GL}(n, \mathbb{F})$.

Theorem 4.10. *Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is as in Lemma 4.9, and that G is completely reducible. Then G is nilpotent if and only if $\psi_\varrho(G)$ is nilpotent and $G_\varrho \leq \mathsf{Z}(G)$.*

Proof. One direction is elementary, and the other is Lemma 4.9. □

Theorem 4.10 transforms nilpotency testing of a finitely generated completely reducible subgroup G of $\mathrm{GL}(n, \mathbb{F})$ into an equivalent pair of problems: testing whether $G_\varrho \leq \mathsf{Z}(G)$, and testing whether $\psi_\varrho(G)$ is nilpotent. If ϱ is a maximal ideal of Δ then Δ/ϱ is a finite field, and we can test nilpotency of $\psi_\varrho(G)$ as in Detinko and Flannery (2006a). To test whether $G_\varrho \leq \mathsf{Z}(G)$ we need a generating set for G_ϱ. This may be achieved if in addition to the input generating set $\{g_1, \ldots, g_r\}$ for G, we know either (i) a transversal for the cosets of G_ϱ in G, or (ii) a presentation for $G/G_\varrho \cong \psi_\varrho(G)$. In case (i), as long as the index $|G : G_\varrho| = |\psi_\varrho(G)|$ is not too large then the Schreier method (Holt et al., 2005, Section 2.5, pp. 41–45) is a realistic option for finding a generating set of G_ϱ. In case (ii), suppose that each relator w_j in the known presentation of $\psi_\varrho(G)$ is written as a word in the $\psi_\varrho(g_i)$. Then by replacing each occurrence of $\psi_\varrho(g_i)$ in w_j by g_i, $1 \leq i \leq r$, we get a generating set for a subgroup of G whose normal closure in G is G_ϱ. (This is the 'normal subgroup generators' method; cf. Holt et al. (2005, pp. 299–300).) As a consequence, the following lemma solves the problem of testing whether G_ϱ is central in G.

Lemma 4.11. *Let $G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, \mathbb{F})$ and*

$$\psi_\varrho(G) = \langle \psi_\varrho(g_1), \ldots, \psi_\varrho(g_r) \mid w_1(\psi_\varrho(g_i)), \ldots, w_s(\psi_\varrho(g_i)) \rangle.$$

Then G_ϱ is the normal closure in G of the subgroup

$$\widetilde{G_\varrho} = \langle w_1(g_i), \ldots, w_s(g_i) \rangle.$$

Hence $G_\varrho \leq \mathsf{Z}(G)$ if and only if $w_j(g_i) \in \mathsf{Z}(G)$ for all j, $1 \leq j \leq s$, in which case $G_\varrho = \widetilde{G_\varrho}$.

Later subsections address the issue of finding a presentation of $\psi_\varrho(G)$.

4.3. Deciding finiteness

After nilpotency testing of $G \leq \mathrm{GL}(n, \mathbb{F})$, we can move on to tackling other basic computational problems for G, such as testing whether G is finite.

Deciding finiteness of matrix groups over algebraic number fields and functional fields was considered by various authors, and a practical implementation was obtained by Beals in GAP for groups over $\mathbb{Q}$ (see Babai et al. (1993)). The method for deciding finiteness that we introduce in this subsection is a general approach to the problem that is uniform with respect to the ground field. We apply it here only for nilpotent groups, while the general case is part of a separate research.

Let $\operatorname{char}\mathbb{F} = 0$, and let ϱ be an ideal of the subring Δ of $\mathbb{F}$ such that Δ/ϱ is finite. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free. Then, obviously, $G \leq \mathrm{GL}(n, \Delta)$ is finite if and only if G_ϱ is trivial. This suggests a very simple and general finiteness test for G. However, efficiency of this test depends on knowing an efficient method to decide whether G_ϱ is trivial. If G is nilpotent then we have such a method by Lemma 4.11.

`IsNilpotentFinite`(G)

Input: A nilpotent subgroup $G = \langle g_1, \ldots, g_r \rangle$ of $\mathrm{GL}(n, \mathbb{F})$, $\operatorname{char}\mathbb{F} = 0$.
Output: a message 'true' meaning that G is finite; and 'false' otherwise.

if $G_u \neq 1$
  then return 'false';
else if $G_\varrho \neq 1$
  then return 'false';
else return 'true'.

Once G is confirmed to be finite, then we know that $|G| = |\psi_\varrho(G)|$. Computing the order of G is thus reduced to the order problem for matrix groups over a finite field. Testing whether $G_\varrho \neq 1$ in `IsNilpotentFinite`(G) is viable by Lemma 4.11, because it is possible to compute efficiently a presentation for the nilpotent group $\psi_\varrho(G)$ over a finite field.

Now we consider $\mathbb{F}$ of positive characteristic.

Lemma 4.12. *Let $G = \langle g_1, \ldots, g_r \rangle$ be a nilpotent subgroup of $\mathrm{GL}(n, \mathbb{F})$, $\operatorname{char}\mathbb{F} > 0$. Then G_u is finite.*

Proof. Schur's First Theorem (Suprunenko, 1976, p. 181) asserts that a periodic subgroup of $\mathrm{GL}(n, \mathbb{F})$ is locally finite. As G is nilpotent, G_u is unipotent and so periodic. Then the result follows, because $G_u = \langle (g_1)_u, \ldots, (g_r)_u \rangle$ is finitely generated. □

By Lemmas 4.1 and 4.12, if G is nilpotent then G is finite precisely when G_s is finite. Let $\mathbb{F}$ be perfect, and suppose that all torsion elements of $\mathcal{G}(n, \Delta, \varrho)$ are unipotent. Then as $G_s/(G_s)_\varrho$ is finite, G is finite if and only if $(G_s)_\varrho$ is trivial, by Lemma 4.9. If $\mathbb{F}$ is not perfect then we can still find a normal unipotent subgroup U of G such that G/U is isomorphic to a completely reducible subgroup of $\mathrm{GL}(n, \mathbb{F})$ (see the discussion at the end of Section 4.1), and the above reasoning goes through again.

For another method to decide finiteness of G, that can be incorporated with nilpotency testing of G, see Section 4.5.

4.4. Polycyclic presentations

A finitely generated nilpotent group is polycyclic, and therefore has a (consistent) polycyclic presentation. One major benefit of possessing a polycyclic presentation for a nilpotent subgroup G of $\mathrm{GL}(n, \mathbb{F})$ is that we gain access to the numerous existing algorithms for abstract polycyclic groups (see Sims (1994, Chapter 9), Holt et al. (2005, Chapter 8), and the package 'Polycyclic' in GAP (The GAP group, 2006)), which may be used to further investigate the structure of G.

The papers Assmann and Eick (2005, 2007) deal with the problem of constructing a polycyclic presentation for a finitely generated subgroup G of $\mathrm{GL}(n, \mathbb{Q})$. Specifically, the algorithm `PolycyclicPresentation`(G) in Assmann and Eick (2005) attempts to compute polycyclic presentations for $\psi_\varrho(G)$, G_ϱ/U_ϱ, and U_ϱ, where $\varrho = p\mathbb{Z}_\pi$ for a finite set π of primes not

containing the odd prime p, $\psi_\varrho : \mathrm{GL}(n, \mathbb{Z}_\pi) \to \mathrm{GL}(n, p)$ is the associated congruence homomorphism, and U_ϱ is a unipotent radical of G_ϱ. If G is polycyclic then the algorithm returns a polycyclic presentation of G. The algorithm fails to terminate if G is solvable but not polycyclic (i.e. U_ϱ is not finitely generated). In this subsection we propose a modification of `PolycyclicPresentation` which either returns a polycyclic presentation of G, or detects that G is not nilpotent.

The paper Assmann and Eick (2007) contains another algorithm, `IsPolycyclic`(G), for polycyclicity testing of a subgroup G of $\mathrm{GL}(n, \mathbb{Q})$. `IsPolycyclic`$(G)$ always terminates, returning either a polycyclic presentation for G, or a message that G is not polycyclic. A nilpotency testing algorithm based on `IsPolycyclic`(G) is also given in Assmann and Eick (2007). That algorithm has the following stages: (i) testing whether G is polycyclic, (ii) testing whether G/U is nilpotent, where U is a unipotent radical of G, and (iii) testing whether G acts nilpotently on U. Our approach in this subsection avoids the possibly time-consuming step (i), and replaces step (iii) with a simpler test.

The strategy of our algorithm is as follows. Let G be a finitely generated subgroup of $\mathrm{GL}(n, \mathbb{F})$, where for convenience $\mathbb{F}$ is assumed to be perfect. After applying `Reduction`(G), we will know either that G is not nilpotent, or that $G \leq \langle G_u, G_s \rangle$, G_u is unipotent, and $[G_u, G_s] = 1$. In the latter event, the problem splits into two: finding polycyclic presentations for G_u and G_s. Note that if we have to proceed further after `Reduction`(G), then the finitely generated nilpotent group $G_u \leq \mathrm{UT}(n, \mathbb{F})$ is definitely polycyclic. Next, we apply a congruence homomorphism ψ_ϱ to G_s, thereby again splitting the problem for G_s into two: finding presentations for the image $\psi_\varrho(G_s)$ and the kernel $(G_s)_\varrho$ of ψ_ϱ on G_s. Of course, ϱ is chosen so that a solution of the former problem is known; for example, $\psi_\varrho(G_s)$ is over a finite field. We recall that if G is nilpotent then $(G_s)_\varrho \leq \mathsf{Z}(G_s)$ (see Lemma 4.9); i.e. $(G_s)_\varrho$ is abelian.

`PresentationNilpotent`(G)

Input: $G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, \mathbb{F})$, $\mathbb{F}$ perfect.
Output: a polycyclic presentation of G, or a message 'false' meaning that G is not nilpotent.

(1) If `Reduction`(G) = 'false' then return 'false'; else go to step (2).
(2) Determine a polycyclic presentation of G_u as a finitely generated subgroup of $\mathrm{UT}(n, R)$, R a finitely generated subring of $\mathbb{F}$.
(3) Compute a generating set for $\psi_\varrho(G_s)$, and use this to attempt to construct a polycyclic presentation of $\psi_\varrho(G_s)$. Return 'false' if the attempt fails.
(4) Determine a generating set for $(G_s)_\varrho$. If $(G_s)_\varrho$ is not central in G_s then return 'false'. Else construct a polycyclic presentation of the finitely generated abelian group $(G_s)_\varrho$.
(5) Combine the presentations of $\psi_\varrho(G_s)$ and $(G_s)_\varrho$ found in steps (3) and (4) to get a polycyclic presentation of G_s.
(6) Combine the presentations of G_u and G_s found in steps (2) and (5) to get a polycyclic presentation of $G^* = G_u G_s$ and thence a polycyclic presentation of $G \leq G^*$.

Implementation of `PresentationNilpotent` depends on the availability of algorithms for computing the polycyclic presentations in steps (2) and (4). Such algorithms are presently available for finite fields and number fields (see Assmann and Eick (2005, 2007)).

4.5. Testing nilpotency using an abelian series; the adjoint representation

Methods for testing nilpotency of matrix groups, relying on properties of nilpotent linear groups, were proposed in Detinko and Flannery (2006a). Although those methods were applied only to groups over finite fields, they are valid over other fields as well. In this subsection we justify this statement.

As in Detinko and Flannery (2006a, Subsection 2.2) we define a recursive procedure `SecondCentralElement`(G, H) which accepts as input finitely generated subgroups G, H of $\mathrm{GL}(n, \mathbb{F})$, $\mathbb{F}$ any field, where H is a non-abelian normal subgroup of G. If G is nilpotent then the recursion terminates in a number of rounds no greater than the nilpotency class of G, returning an element of $Z_2(H) \setminus \mathsf{Z}(H)$. We therefore seek an upper bound on nilpotency class of nilpotent subgroups of $\mathrm{GL}(n, \mathbb{F})$. (Such a bound exists only for certain fields $\mathbb{F}$. For instance, if $\mathbb{F}$ is algebraically closed then $\mathrm{GL}(n, \mathbb{F})$ contains nilpotent groups of every class; see Suprunenko (1976, Corollary 1, p. 214).) Application of a suitable congruence homomorphism may provide a bound as required.

Lemma 4.13. *Let G be a nilpotent completely reducible subgroup of $\mathrm{GL}(n, \mathbb{F})$ contained in $\mathrm{GL}(n, \Delta)$, Δ a finitely generated subring of $\mathbb{F}$. Let ϱ be an ideal of Δ as in Lemma 4.9. Then the nilpotency class of G is at most the nilpotency class of $\psi_\varrho(G)$ plus one.*

Proof. This is clear by Lemma 4.9. □

Example 4.14. Theorem 2 of Wehrfritz (2001) gives an upper bound of $3n/2$ for the nilpotency class of subgroups of $\mathrm{GL}(n, \mathbb{Q})$. This further implies an upper bound of $3mn/2$ for subgroups of $\mathrm{GL}(n, \mathbb{P})$, where $\mathbb{P}$ is a number field of degree m over $\mathbb{Q}$. Suppose that G is a finitely generated nilpotent subgroup of $\mathrm{GL}(n, \mathbb{Q}(X))$. Since $\mathrm{UT}(n, \mathbb{F})$ has nilpotency class $n-1$ (see Suprunenko (1976, Theorem 13.5, p. 89)), it follows from Lemmas 4.1 and 4.13 that the nilpotency class of G is at most $\frac{3n}{2} + 1$. Similar remarks apply to groups over $\mathbb{Q}(X_1, \ldots, X_m)$.

Example 4.15. Let q be a power of a prime p. If G is a finitely generated nilpotent subgroup of $\mathrm{GL}(n, \mathbb{F})$ for $\mathbb{F} = \mathrm{GF}(q)(X)$ then G has nilpotency class at most $l_{n,q} + 1$, where $l_{n,q}$ is an upper bound on the class of nilpotent subgroups of $\mathrm{GL}(n, q)$. A formula for $l_{n,q}$ may be deduced from Bialostocki (1986, Theorem C.3):

$$l_{n,q} = n \cdot \max\{(t-1)s + 1 \mid t \neq p \text{ prime}, t \leq n, t^s \text{dividing } q - 1\}. \tag{3}$$

That is, $n((t-1)s+1)$ is an upper bound on the class of a Sylow t-subgroup of $\mathrm{GL}(n, q)$, where t^s is the largest power of the prime t dividing $q-1$ (slightly better bounds are known for special cases e.g. $t = 2$). We restrict to $t \leq n$ in (3) because a t-subgroup of $\mathrm{GL}(n, q)$ is abelian if $t \neq p$ and $t > n$ (cf. Detinko and Flannery (2006a, Lemma 2.25)).

We assume henceforth that we are able to specify a number $k_\mathbb{F}$ such that if termination does not occur in $k_\mathbb{F}$ rounds or less then `SecondCentralElement`(G, H) reports that G is not nilpotent; otherwise, the procedure returns an element $a \in Z_2(H) \setminus \mathsf{Z}(H)$ such that $[G, a] \leq \mathsf{Z}(H)$.

Other procedures in Detinko and Flannery (2006a) that were originally designed for finite fields $\mathbb{F}$ also carry over to any $\mathbb{F}$. Given $a \in Z_2(G) \setminus \mathsf{Z}(G)$, let $\varphi_a : G \to \mathsf{Z}(G) \cap [G, G]$ be the homomorphism defined by $g \in G \mapsto [g, a]$. If G is completely reducible then `NonCentralAbelian`(G, a) returns the abelian normal subgroup $A = \langle a \rangle^G = \langle a, \varphi_a(G) \rangle$ of G, and `Centralizer`(G, A) returns a generating set for the kernel $\mathsf{C}_G(A)$ of φ_a. `NonCentralAbelian`(G, a) requires a 'cutting procedure' for the enveloping algebra $\langle A \rangle_\mathbb{F}$, to

reduce computations to the case of cyclic $\varphi_a(G)$. Also note that Detinko and Flannery (2006a, Lemma 2.17 and Corollary 2.18) hold for any field $\mathbb{F}$; so that, as in the finite field case, we get a moderate upper bound on the index $|G : \mathsf{C}_G(A)|$.

The cutting procedure described in Rónyai (1993, Section 3) finds the simple components of a finite-dimensional commutative semisimple algebra over any field $\mathbb{F}$, input by a set of algebra generators. When $\mathbb{F} = \mathbb{Q}$ another method, based on Dixon (1985, Lemma 5), can be applied (see Assmann and Eick (2005, Section 5.2)). The main requirement here is an efficient method for factorizing polynomials over $\mathbb{F}$.

The discussion above shows that the recursive procedure `TestSeries` of Detinko and Flannery (2006a, Subsection 2.4) can be defined over any field $\mathbb{F}$. The basic steps in the recursion are outlined in Detinko and Flannery (2006a, pp. 113–114). If it does not detect that the input finitely generated subgroup G of $\mathrm{GL}(n, \mathbb{F})$ is not nilpotent, then `TestSeries`(G, l) returns a series

$$\langle 1_n \rangle \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_l \trianglelefteq C_l \lhd \cdots \lhd C_2 \lhd C_1 \lhd G \tag{4}$$

where the A_i are abelian, the normal subgroup C_i of G is the centralizer of A_i in C_{i-1}, and the factors C_{i-1}/C_i are abelian. That is, all factors of consecutive terms in (4) are abelian, except possibly the middle factor C_l/A_l. The construction of further terms in (4) can continue, with strict inclusions everywhere except possibly in the middle of the series, as long as C_l is non-abelian.

Lemma 4.16. *For some $l \leq n-1$, the term C_l in (4) is abelian.*

Proof. Cf. the proof of Detinko and Flannery (2006a, Lemma 2.20). □

N.B. *Until further notice in this subsection, G is completely reducible.*

Lemma 4.17. $\mathsf{Z}(G)$ *is contained in every term C_i of the series (4). Therefore, if G is nilpotent, then G/C_l is finite.*

Proof. Certainly $\mathsf{Z}(G) \leq C_1 = \mathsf{C}_G(A)$. Assume that $\mathsf{Z}(G) \leq C_{k-1}$; then $\mathsf{Z}(G)$ is contained in the C_{k-1}-centralizer C_k of A_k. Then the second statement is clear by Lemma 4.6. □

Corollary 4.18. *Suppose that G is nilpotent. Then G is finite if and only if C_l in (4) is finite.*

Corollary 4.18 gives another finiteness test for completely reducible nilpotent subgroups G of $\mathrm{GL}(n, \mathbb{F})$; cf. Section 4.3. This test requires that we are able to decide finiteness of the finitely generated completely reducible abelian matrix group C_l. To that end, the next result may be useful.

Lemma 4.19. *If G is non-abelian nilpotent then C_l has non-trivial torsion.*

Proof. Suppose that C_l is torsion-free. Let $a \in Z_2(G) \setminus \mathsf{Z}(G)$. Since $a^m \in \mathsf{Z}(G)$ for some m by Lemma 4.6, there exists $g \in G$ such that $[g, a] \in \mathsf{Z}(G)$ has finite non-trivial order (dividing m). This contradicts $[g, a] \in A_1 \leq C_l$. □

Suppose now that G is finite. Then we can apply Detinko and Flannery (2006a, Lemma 2.23) to G. That is, we refine (4) to a polycyclic series of G, then test nilpotency of G via prime factorization of the cyclic quotients in the refined series, and checking that factors for different primes commute. Hence the algorithm `IsNilpotent` from Detinko and Flannery (2006a, Section 2) can be employed for nilpotency testing of G. In the more general setting we label this algorithm `IsFiniteNilpotent`. This algorithm, which accepts only finite $G \leq \mathrm{GL}(n, \mathbb{F})$

as input, also yields the Sylow decomposition of nilpotent G. Complexity analysis (in terms of number of field operations) of `IsFiniteNilpotent` is undertaken in Detinko and Flannery (2006a, Section 2).

Now we examine the case that G is infinite. First we state a few structural results.

Lemma 4.20. *Let π be the set of primes less than or equal to n. Suppose that G is nilpotent. Then every element of (the finite group) $G/\mathsf{Z}(G)$ has order divisible only by the primes in π. Moreover, no element of $G/\mathsf{Z}(G)$ has order divisible by* $\mathrm{char}\,\mathbb{F}$.

Proof. It suffices to prove the lemma for irreducible G (cf. Remark 4.7). Proofs of the irreducible case are given in Suprunenko (1976, Chapter 7); see Corollary 1, p. 206, and Theorem 2, p. 216, of Suprunenko (1976). □

Corollary 4.21 (*Cf. Example* 4.15). *If G is nilpotent then for all primes $p > n$, a Sylow p-subgroup of G is central.*

Recall that a group H is said to be *p-primary*, for a prime p, if $H/\mathsf{Z}(H)$ is a p-group.

Lemma 4.22. *If G is nilpotent then G is a product of p-primary groups for $p \leq n$.*

Now let G be any subgroup of $\mathrm{GL}(n, \mathbb{F})$. Set m to be the $\mathbb{F}$-dimension of the enveloping algebra $\langle G\rangle_{\mathbb{F}}$. Define the adjoint representation $\mathrm{adj} : G \to \mathrm{GL}(m, \mathbb{F})$ by $\mathrm{adj}(g) : x \mapsto gxg^{-1}$, $x \in \langle G\rangle_{\mathbb{F}}$. Clearly $\ker \mathrm{adj} = \mathsf{Z}(G)$. If G is nilpotent and completely reducible then $\mathrm{adj}(G)$ is a finite completely reducible subgroup of $\mathrm{GL}(m, \mathbb{F})$, by Lemma 4.20 and Maschke's theorem. If $G = \langle g_1, \ldots, g_r\rangle$ then by Beals (1999, Lemma 4.1) we can construct a basis of $\langle G\rangle_{\mathbb{F}}$ as a straight-line program of length m over $\{g_1, \ldots, g_r\}$. Knowing a basis of $\langle G\rangle_{\mathbb{F}}$ we can calculate $\mathrm{adj}(G)$ by solving a system of linear equations. The adjoint representation is another way of transferring nilpotency testing to the case of finite groups.

The results presented so far in this subsection lead to the following algorithm to test nilpotency of G using an abelian series and the adjoint representation. The input generators of G are diagonalizable, but G cannot be assumed in advance to be completely reducible. Also, as mentioned earlier, this algorithm requires knowledge of an upper bound on nilpotency class of nilpotent subgroups of $\mathrm{GL}(n, \mathbb{F})$.

`IsNilpotentAdjoint`(G)

Input: $G = \langle g_1, \ldots, g_r\rangle \leq \mathrm{GL}(n, \mathbb{F})$, $g_i \in \mathrm{GL}(n, \mathbb{F})$ diagonalizable.
Output: a message 'true' meaning that G is nilpotent, or a message 'false' meaning that G is not nilpotent.

for $i \in [1\,..\,r]$
  do $\bar{g}_i := \mathrm{adj}(g_i)$;
$\bar{G} := \langle \bar{g}_1, \ldots, \bar{g}_r\rangle$;
if $(\bar{g}_i)_u \neq 1$ for some i
  then return 'false';
else invoke `TestSeries`$(\bar{G})$;
if $\bar{G}$ is infinite
  then return 'false';
else invoke `IsFiniteNilpotent`$(\bar{G})$.

For testing whether $\bar{G}$ is infinite after invoking `TestSeries`$(\bar{G})$, see Corollary 4.18.

Parts of `IsNilpotentAdjoint` that use polynomial factorization (e.g. the cutting procedure) have running time dependent on the coefficient field. Also, computation of $\bar{G} = \mathrm{adj}(G)$ entails squaring the dimension in worst-case; so may be time-consuming and efficient only for small n.

If `IsNilpotentAdjoint`(G) returns 'true' then the algorithm furnishes additional information about the input group G, such as its decomposition into p-primary subgroups. Also, knowing a generating set for $\bar{G}$ we can find a generating set for $\mathsf{Z}(G) = \ker \mathrm{adj}$ (by the Schreier method, or using a presentation of $\bar{G} \cong G/\mathsf{Z}(G)$ to pull back to 'normal subgroup generators', hence a generating set, of $\mathsf{Z}(G)$; see before Lemma 4.11). If we can find a polycyclic presentation of the finitely generated completely reducible abelian matrix group $\mathsf{Z}(G)$, then this can be combined with a polycyclic presentation of $G/\mathsf{Z}(G)$. Thus we gain one more method for constructing a polycyclic presentation of G.

4.6. *Nilpotency testing via change of ground domain and abelian series*

Finally, we outline the simplest and most effective combination of our ideas for nilpotency testing of finitely generated matrix groups, over a perfect field $\mathbb{F}$. This is the version of nilpotency testing used in the GAP package 'Nilmat' (Detinko et al., 2007).

4.6.1. *The algorithm*

`IsNilpotentMatGroup` as set out below tests nilpotency over an infinite field $\mathbb{F}$, via `Reduction`(G) (if $\mathbb{F}$ is perfect), and applying a congruence homomorphism ψ_ϱ to G_s, where ϱ satisfies the hypotheses of Lemma 4.9. Nilpotency of $\psi_\varrho(G_s)$ is tested using an abelian series (4) of $\psi_\varrho(G_s)$ in $\mathrm{GL}(n, q)$: see Section 4.5. Note that Section 4.5 was written for input groups over any (perfect) field, and hence is applicable to $\psi_\varrho(G_s) \leq \mathrm{GL}(n, q)$. If G is nilpotent then $(G_s)_\varrho \leq \mathsf{Z}(G_s)$, and this containment can be tested via Lemma 4.11.

`IsNilpotentMatGroup`(G)

Input: $G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, \mathbb{F})$.
Output: a message 'true' meaning that G is nilpotent, or a message 'false' meaning that G is not nilpotent.

(1) `Reduction`(G).
(2) Construct $\psi_\varrho(G_s) \leq \mathrm{GL}(n, q)$.
(3) Test nilpotency of $\psi_\varrho(G_s)$.
(4) Test whether $(G_s)_\varrho \leq \mathsf{Z}(G_s)$.

There are several advantages of the approach embodied in `IsNilpotentMatGroup`. Firstly, by reducing the amount of computation over the original field $\mathbb{F}$, we hope to escape unfortunate consequences which sometimes occur when computing over infinite fields (e.g. blow-up of size of matrix entries). Another issue relates to upper bounds on nilpotency class. If G is nilpotent then procedures used in `TestSeries` to construct the series (4) that depend on a class bound for the potentially nilpotent group $\psi_\varrho(G)$, such as `SecondCentralElement`, are guaranteed to terminate more quickly than for arbitrary nilpotent subgroups of $\mathrm{GL}(n, q)$. For example, if $\mathbb{F} = \mathbb{Q}$ then $\psi_\varrho(G)$ inherits from $G \leq \mathrm{GL}(n, \mathbb{Q})$ an upper bound $3n/2$ on nilpotency class; this can be compared with the general bound (3) for $\mathrm{GL}(n, q)$ stated in Example 4.15.

It is desirable to retain complete reducibility in step (2) of `IsNilpotentMatGroup`. That is, the Jordan decomposition over the top field $\mathbb{F}$ is unavoidable; we seek not to repeat it in

Table 1
Sample running times for nilpotency testing algorithms

Group	Degree	Field	No. of generators	Runtime
G_1	9	5^6	6	0 : 00 : 26.890
G_2	127	2^7	3	0 : 18 : 37.092
G_3	12	5^6	9	0 : 12 : 35.592
G_4	30	11^4	9	0 : 17 : 39.749
G_5	63	2^6	11	0 : 18 : 25.842
G_6	90	2^8	54	0 : 21 : 28.280
G_7	96	5^4	63	0 : 35 : 27.546
G_8	120	11^4	27	1 : 07 : 17.702
G_9	100	$\mathbb{Q}$	12	0 : 08 : 39.641
G_{10}	200	$\mathbb{Q}$	27	0 : 09 : 16.344
G_{11}	128	$\mathbb{Q}$	93	0 : 14 : 07.398
G_{12}	150	13^3	2	0 : 00 : 23.688
G_{13}	350	$\mathbb{Q}$	4	0 : 00 : 55.047
G_{14}	25	$\mathbb{Q}$	13	0 : 16 : 25.859

$\mathrm{GL}(n, q)$. Let q be a power of the prime p. In order for the input $\psi_\varrho((g_i)_s)$ to `TestSeries` to be diagonalizable, these elements must all be of order coprime to p. Equivalently ϱ should be chosen so that $f_i(X)$ and $f_i'(X)$ are coprime for all i, where $f_i(X)$ is the minimal polynomial of $\psi_\varrho((g_i)_s)$ (note that $f_i(X)$ is the image $\psi_\varrho(h_i(X))$ of the minimal polynomial $h_i(X)$ of g_i). Selection of ϱ is a number theory problem if $\mathbb{F}$ is a number field.

Lemma 4.23. *If G_s is nilpotent and $p > n$ then any preimage of $(\psi_\varrho(G_s))_u$ in G_s is central.*

Proof. Let $g \in G_s$. Then $\psi_\varrho(g)_u = \psi_\varrho(g^l)$ for some l, and $\psi_\varrho(g^{lp^k}) = 1$ for some k. That is, $g^{lp^k} \in (G_s)_\varrho \leq \mathsf{Z}(G_s)$. Then by Corollary 4.21, $g^l \in \mathsf{Z}(G_s)$. □

Lemma 4.23 indicates that we may reasonably expect $\psi_\varrho(G_s)$ to be completely reducible if ϱ is chosen so that $p > n$. However, if n is large then of course it is advisable to work with $p \leq n$.

4.6.2. Implementation and experimental results

Our implementation of `IsNilpotentMatGroup` for groups defined over $\mathbb{Q}$ also includes an algorithm `IsNilpotentMatGroupFF` for testing nilpotency over finite fields, according to Section 4.5. To construct congruence subgroups, `IsNilpotentMatGroup` uses some functions from the GAP package 'Polenta' (Assmann and Eick, 2007).

Table 1 illustrates performance of `IsNilpotentMatGroup` for various input parameters: degree; size of the field if finite, or size of generator entries if the field is $\mathbb{Q}$; and number of generators. The last column of Table 1 gives CPU time in the format minutes : seconds : milliseconds. The computations were done on a Pentium 4 with 1.73 GHz under Windows, using GAP 4. The standard GAP function `IsNilpotent` failed for all groups in Table 1.

As one might expect, the most challenging input groups are the nilpotent groups, because for them all stages of the algorithms have been passed through. On the other hand, if the input is not nilpotent, then this is confirmed very quickly. For example, if the input does not have an abelian

series – in particular, if it is not solvable – then the algorithm terminates at the `TestSeries` stage (see Section 4.5).

Thus, for proper testing of our algorithms, we need an extensive set of examples of nilpotent matrix groups. Constructing special classes of nilpotent matrix groups is a problem of interest in its own right. We have implemented an algorithm, `MaximalAbsolutelyIrreducibleNilpotentMatGroup`(n, p, l), that constructs absolutely irreducible maximal nilpotent subgroups of $\mathrm{GL}(n, p^l)$, for input degree n and field size p^l. If r divides $p^l - 1$ for each prime divisor r of n then such a subgroup of $\mathrm{GL}(n, p^l)$ is unique up to conjugacy; otherwise, such subgroups of $\mathrm{GL}(n, p^l)$ do not exist (see Suprunenko (1976, Chapter 7)). If $n = r^a$ and the prime r divides $p^l - 1$ then `MaximalAbsolutelyIrreducibleNilpotentMatGroup`(n, p, l) returns the group generated by a Sylow r-subgroup of $\mathrm{GL}(n, p^l)$, and all scalars. For other degrees n, this algorithm returns the group generated by all scalars, and a Kronecker product of Sylow r_i-subgroups of $\mathrm{GL}(r_i^{a_i}, p^l)$, $n = \prod_{i=1}^{k} r_i^{a_i}$.

To check steps which rely on the Jordan decomposition, we implemented another procedure, `ReducibleNilpotentMatGroup`. This procedure returns reducible but not completely reducible nilpotent groups over finite fields and over $\mathbb{Q}$.

The groups G_i in Table 1 for $i \leq 5$ are absolutely irreducible nilpotent groups constructed by `MaximalAbsolutelyIrreducibleNilpotentMatGroup`. The reducible groups G_6, G_7, G_8, and G_9, G_{10}, G_{11}, are constructed by `ReducibleNilpotentMatGroup`. Finally, G_{12}, G_{13}, and G_{14} are non-nilpotent groups; $G_{12} = \mathrm{GL}(150, 13^3)$, $G_{13} = \mathrm{GL}(350, \mathbb{Z})$, and G_{14} is the group `POL_PolExamples2(40)` from Polenta, an infinite solvable subgroup of $\mathrm{GL}(25, \mathbb{Q})$.

'Nilmat' contains a variety of other functions for computing with nilpotent matrix groups. These include functions for deciding finiteness, computing orders of finite nilpotent groups, finding the Sylow system of a nilpotent group over a finite field, and testing whether a nilpotent group is completely reducible. These functions are by-products of nilpotency testing, and in many cases run much faster than the corresponding GAP functions. Additionally 'Nilmat' contains a library of the nilpotent primitive groups over finite fields (based on Detinko and Flannery (2004)).

Acknowledgements

We are immensely indebted to Professor Bettina Eick for her hospitality during our visits to Technische Universität Braunschweig, for fruitful discussions then and afterwards, and moreover for her very generous assistance to us in writing a GAP implementation of some of the ideas in this paper.

This publication has emanated from research conducted with the financial support of Science Foundation Ireland. The first author was supported by Deutscher Akademischer aus Tauschdienst (DAAD) grant A/06/32418.

References

Assmann, B., 2003. Polycyclic presentations for matrix groups. Diplomarbeit. Technische Universität Braunschweig.

Assmann, B., Eick, B., 2005. Computing polycyclic presentations for polycyclic rational matrix groups. J. Symbolic Comput. 40 (6), 1269–1284.

Assmann, B., Eick, B., 2007a. Polenta—Polycyclic presentations for matrix groups. A refereed GAP 4 package. See http://www.gap-system.org/Packages/polenta.html.

Assmann, B., Eick, B., 2007b. Testing polycyclicity of finitely generated rational matrix groups. Math. Comput. 76, 1669–1682.

Babai, L., Beals, R., Cai, J., Ivanyos, G., Luks, E.M., 1996. Multiplicative equations over commuting matrices. In: Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms. Atlanta, GA, 1996. ACM, New York, pp. 498–507.

Babai, L., Beals, R., Rockmore, D.N., 1993. Deciding finiteness of matrix groups in deterministic polynomial time. In: Proc. of International Symposium on Symbolic and Algebraic Computation ISSAC '93. ACM Press, pp. 117–126.

Beals, R., 1999. Algorithms for matrix groups and the Tits alternative. J. Comput. Syst. Sci. 58, 260–279.

Bialostocki, A., 1986. The nilpotency class of the p-Sylow subgroups of GL(n, q) where $(p, q) = 1$. Canad. Math. Bull. 29 (2), 218–223.

Cohn, P.M., 1989. Algebra, vol. 2, second edn. John Wiley & Sons Ltd., Chichester.

Detinko, A.S., Eick, B., Flannery, D.L., 2007. Nilmat—Computing with nilpotent matrix groups. A refereed GAP 4 package. See http://www.gap-system.org/Packages/nilmat.html.

Detinko, A.S., Flannery, D.L., 2004. Classification of nilpotent primitive linear groups over finite fields. Glasgow Math. J. 46, 585–594.

Detinko, A.S., Flannery, D.L., 2005. Locally nilpotent linear groups. Irish Math. Soc. Bull. (56), 37–51.

Detinko, A.S., Flannery, D.L., 2006a. Computing in nilpotent matrix groups. LMS J. Comput. Math. 9, 104–134 electronic.

Detinko, A.S., Flannery, D.L., 2006b. Corrigendum to locally nilpotent linear groups [Irish Math. Soc. Bull. (56) (2005) 37–51; mr2232095]. Irish Math. Soc. Bull. (57), 103.

Dixon, J.D., 1971. The Structure of Linear Groups. Van Nostrand Reinhold, London.

Dixon, J.D., 1985. The orbit-stabilizer problem for linear groups. Canad. J. Math. 37 (2), 238–259.

Eick, B., 2005. Computational group theory. Jahresbericht der DMV 107 Heft 3, pp. 155–170.

The GAP group (2006). GAP - Groups, Algorithms, and Programming, Version 4.4.9, http://www.gap-system.org.

Holt, D.F., Eick, B., O'Brien, E.A., 2005. Handbook of Computational Group Theory. Chapman & Hall/CRC Press, Boca Raton, London, New York, Washington.

Lo, E.H., 1998. Finding intersections and normalisers in finitely generated nilpotent groups. J. Symbolic Comput. 25, 45–59.

Lo, E.H., Ostheimer, G., 1999. A practical algorithm for finding matrix representations for polycyclic groups. J. Symbolic Comput. 28 (3), 339–360.

Luks, E.M., 1992. Computing in solvable matrix groups. In: Proc. 33rd IEEE Symposium on Foundations of Computer Science. pp. 111–120.

Ostheimer, G., 1999. Practical algorithms for polycyclic matrix groups. J. Symbolic Comput. 28, 361–379.

Rónyai, L., 1993. Computations in associative algebras. In: DIMACS Series in Discrete Mathematics, vol. 11. pp. 221–243.

Segal, D., 1983. Polycyclic Groups. Cambridge University Press, Cambridge.

Sims, C.C., 1994. Computation with finitely presented groups. In: Encyclopedia of Mathematics and its Applications, vol. 48. Cambridge University Press, New York.

Suprunenko, D.A., 1976. Matrix Groups. In: Transl. Math. Monogr., vol. 45. American Mathematical Society, Providence, RI.

Wehrfritz, B.A.F., 1973. Infinite Linear Groups. Springer-Verlag, Berlin, Heidelberg, New York.

Wehrfritz, B.A.F., 2001. Nilpotent subgroups of GL$(n, \mathbb{Q})$. Glasgow. Math. J. 43 (3), 477–485.