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# Periodic subgroups of projective linear groups in positive characteristic

**Research Article**

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**Abstract:** We classify the maximal irreducible periodic subgroups of $\mathrm{PGL}(q,\mathbb{F})$, where $\mathbb{F}$ is a field of positive characteristic $p$ transcendental over its prime subfield, $q \neq p$ is prime, and $\mathbb{F}^{\times}$ has an element of order $q$. That is, we construct a list of irreducible subgroups $G$ of $\mathrm{GL}(q,\mathbb{F})$ containing the centre $\mathbb{F}^{\times}1_q$ of $\mathrm{GL}(q,\mathbb{F})$, such that $G/\mathbb{F}^{\times}1_q$ is a maximal periodic subgroup of $\mathrm{PGL}(q,\mathbb{F})$, and if $H$ is another group of this kind then $H$ is $\mathrm{GL}(q,\mathbb{F})$-conjugate to a group in the list. We give criteria for determining when two listed groups are conjugate, and show that a maximal irreducible periodic subgroup of $\mathrm{PGL}(q,\mathbb{F})$ is self-normalising.

**MSC:** 20H20, 20Exx

**Keywords:** linear group • periodic group • projective general linear group • field • classification



The classification of finite linear (matrix) groups over a field $\mathbb{K}$ is one of the earliest and most fundamental problems in group theory. Periodic groups are a generalisation of finite groups. In many situations it is difficult to classify all periodic subgroups of $\mathrm{GL}(n,\mathbb{K})$, and, if $\mathrm{GL}(n,\mathbb{K})$ is not itself periodic, one attempts instead to describe only the maximal periodic subgroups of $\mathrm{GL}(n,\mathbb{K})$: this approach is recommended since every periodic subgroup of $\mathrm{GL}(n,\mathbb{K})$ is contained in a maximal periodic subgroup. A description of the maximal periodic subgroups of $\mathrm{GL}(n,\mathbb{C})$ is given in [13, Theorem 7, p. 200]. Although the number of conjugacy classes of such groups is finite, a complete classification of them seems beyond reach. In particular, the problem of classifying the primitive maximal periodic subgroups of $\mathrm{GL}(n,\mathbb{C})$ is equivalent to the problem of classifying the primitive maximal finite subgroups of $\mathrm{SL}(n,\mathbb{C})$, the complexity of which is evident from [5]. However, if $\mathbb{K}$ is a field $\mathbb{F}$ of positive characteristic $p$, then the classification problem becomes more tractable. A maximal irreducible periodic subgroup of $\mathrm{GL}(n,\mathbb{F})$ is conjugate to $\mathrm{GL}(n,\mathbb{F}_a)$, where $\mathbb{F}_a$ is the subfield of $\mathbb{F}$ consisting of all elements that are algebraic over the prime subfield $\mathbb{F}_p$ (see [16, Theorem 1], and cf. [15, Theorem 1], [8, 9.23, p. 155]). That result implies ([16, Theorem 2]) that there are only finitely many conjugacy classes of maximal periodic subgroups of $\mathrm{GL}(n,\mathbb{F})$. Several authors [2, 3, 11, 12, 17] have extended results of [16] to other classical groups over $\mathbb{F}$. More recently, [16] has provided theoretical background for the efficient solution of problems in computational group theory, such as deciding finiteness of matrix groups (see [4]).

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In this paper we study maximal periodic subgroups of the projective general linear group $\mathrm{PGL}(n,\mathbb{F})$. If $\mathbb{F}=\mathbb{F}_a$ then $\mathrm{GL}(n,\mathbb{F})$ is periodic and there is nothing to discuss; hence from now on we insist that $\mathbb{F}$ is transcendental over $\mathbb{F}_p$. Despite the apparent connection, the classification problem for $\mathrm{PGL}(n,\mathbb{F})$ has quite a different nature to the problem for $\mathrm{GL}(n,\mathbb{F})$. To appreciate this, compare the descriptions in [10] and [13, Chapter 6] of Sylow subgroups of $\mathrm{PGL}(n,\mathbb{F})$ and $\mathrm{GL}(n,\mathbb{F})$. The study of periodic subgroups of $\mathrm{PGL}(n,\mathbb{F})$ for all $n$ encounters difficulties not present when studying periodic subgroups of $\mathrm{GL}(n,\mathbb{F})$; while the latter groups obviously give rise to periodic subgroups of $\mathrm{PGL}(n,\mathbb{F})$, there exist periodic subgroups of $\mathrm{PGL}(n,\mathbb{F})$ whose preimages in $\mathrm{GL}(n,\mathbb{F})$ are not periodic. The only viable approach to the problem of classifying periodic subgroups of $\mathrm{PGL}(n,\mathbb{F})$ is to first impose some restriction on the degree $n$. For the corresponding problem in $\mathrm{GL}(n,\mathbb{C})$, a strong result along those lines is the classification in [5] of the finite primitive subgroups of $\mathrm{SL}(q,\mathbb{C})$, $q$ prime. Also, in [1, 6], finite irreducible monomial subgroups of $\mathrm{GL}(q,\mathbb{C})$ are classified up to conjugacy. The restriction to prime degree has significant advantages. For example, an irreducible subgroup of $\mathrm{GL}(q,\mathbb{F})$ is either abelian or absolutely irreducible, and it is either monomial or primitive. These are reasons why some important classes of linear groups have been completely classified only in prime degree. We refer here to the long-standing problem of classification of soluble linear groups. Although detailed descriptions of soluble subgroups of $\mathrm{GL}(n,\mathbb{K})$ are available for arbitrary $n$, full classifications have been achieved mainly for irreducible soluble linear groups of prime degree (see e.g. [13, Chapter V]).

In this paper we classify up to conjugacy the irreducible maximal periodic subgroups of $\mathrm{PGL}(q,\mathbb{F})$, $q\neq p$ prime. This serves as a first step towards classification in more general degrees. Furthermore, our result is connected to another important problem in linear group theory: classification of locally nilpotent linear groups. That problem reduces to a partial case of classifying periodic projective linear groups, namely Sylow $p$-subgroups of $\mathrm{PGL}(q^t,\mathbb{K})$, where $\mathbb{K}$ has characteristic not equal to $p$ (see [13, Chapter VII]). Methods developed in this paper have been fruitfully applied in [7] to the investigation of locally nilpotent linear groups.

Denote the natural projection $\mathrm{GL}(q,\mathbb{F})\to\mathrm{PGL}(q,\mathbb{F})$ by $\pi$. A subgroup $\pi(H)$ of $\mathrm{PGL}(q,\mathbb{F})$ will be called irreducible if its preimage $H$ in $\mathrm{GL}(q,\mathbb{F})$ is irreducible. We assume throughout that the multiplicative group $\mathbb{F}^\times$ of $\mathbb{F}$ has an element $\epsilon$ of order $q$. Our list of the maximal irreducible periodic subgroups of $\mathrm{PGL}(q,\mathbb{F})$ is defined before Lemma 12 below. As will be seen, groups in the list are absolutely irreducible, and, except possibly those isomorphic to $\mathrm{PGL}(q,\mathbb{F}_a)$, are soluble.

By Zorn's Lemma, each irreducible periodic subgroup of $\mathrm{PGL}(n,\mathbb{F})$ is contained in some maximal irreducible periodic subgroup of $\mathrm{PGL}(n,\mathbb{F})$. The corresponding statement for $\mathrm{GL}(n,\mathbb{F})$ is noted here for reference in subsequent argument (recall [16, Theorem 1]).

### Theorem 1.

*A periodic irreducible subgroup of $\mathrm{GL}(n,\mathbb{F})$ is conjugate to a subgroup of $\mathrm{GL}(n,\mathbb{F}_a)$.*

We use the following notation. Let $I$ be the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1_{q-1} & 0\end{pmatrix}$ of order $q$. Unless stated otherwise, if $\mathbb{F}^\times\neq(\mathbb{F}^\times)^q$ then $\alpha$ is an element of $\mathbb{F}^\times$ not in $(\mathbb{F}^\times)^q$ (one requirement for the existence of $\alpha$ is that $\mathbb{F}$ not be algebraically closed). Let $a$ be the diagonal matrix $\mathrm{diag}(1_{q-1},\alpha)$, and denote $Ia$ by $I_\alpha$, so $I_\alpha^q=\alpha 1_q$. For any $\beta\in\mathbb{F}^\times$, $X^q-\beta\in\mathbb{F}[X]$ either is irreducible or has a root in $\mathbb{F}$. Hence the characteristic polynomial $X^q-\alpha$ of $I_\alpha$ is irreducible, and the $\mathbb{F}$-enveloping algebra $\Delta_\alpha=\langle I_\alpha\rangle_{\mathbb{F}}$ of $\langle I_\alpha\rangle$ is a field extension of $\mathbb{F}1_q$ of degree $q$. Let $d=\mathrm{diag}(1,\epsilon,\dots,\epsilon^{q-1})$; we readily check that $dI_\alpha d^{-1}=\epsilon I_\alpha$. Then $\sigma: x\mapsto dxd^{-1}$, $x\in\Delta_\alpha$, defines an $\mathbb{F}$-automorphism of $\Delta_\alpha$ of order $q$. The order of $\mathrm{Aut}(\Delta_\alpha/\mathbb{F}1_q)$ divides $q$, and therefore $\Delta_\alpha/\mathbb{F}1_q$ is Galois (indeed, it is a cyclic field extension of $\mathbb{F}1_q$). The $\mathrm{GL}(n,\mathbb{F})$-normaliser $\mathrm{N}(\Delta_\alpha^\times)$ of $\Delta_\alpha^\times$ is $\langle\Delta_\alpha^\times,d\rangle$.

### Lemma 2.

*Suppose that $h\in\mathrm{GL}(q,\mathbb{F})$ and $h^q=\beta 1_q$, $\beta\in\mathbb{F}^\times$. If $\beta=\alpha^r c^q$, $1\le r\le q-1$, then $tht^{-1}=cI_\alpha^r$ for some $t\in\mathrm{GL}(q,\mathbb{F})$.*

***Proof.*** Both $c^{-1}h$ and $I_\alpha^r$ have the same characteristic polynomial $X^q-\alpha^r$, which is $\mathbb{F}$-irreducible; therefore, $c^{-1}h$ and $I_\alpha^r$ are similar. □

### Lemma 3.
*A field extension $\Delta \subseteq \mathrm{Mat}(q, \mathbb{F}_a)$ of $\mathbb{F}_a 1_q$ of degree $q$ is cyclic.*

**Proof.** The extension $\Delta/\mathbb{F}_a 1_q$ is simple: $\Delta = \mathbb{F}_a(h)$, $\langle h \rangle \leq \mathrm{GL}(q, \mathbb{F}_a)$ irreducible. The extension of $\mathbb{F}_p 1_q$ generated by $\mathbb{F}_p 1_q$ and $h$ has degree $q$, so $h \in \mathbb{F}_p(c)$ where $c \in \Delta$ is a root of the polynomial $X^q - \delta 1_q$ for some $\delta \in \mathbb{F}_p^\times \setminus (\mathbb{F}_p^\times)^q$. Hence $\Delta = \mathbb{F}_a(c)$ is cyclic over $\mathbb{F}_a 1_q$. □

### Lemma 4.
*If $\Delta$ is an algebraic extension of $\mathbb{F}_a$ and $\Delta^\times \neq (\Delta^\times)^q$ then $\Delta^\times/(\Delta^\times)^q$ is finite of order $q$.*

**Proof.** Let $\beta, \gamma \in \Delta^\times \setminus (\Delta^\times)^q$. The field $\mathbb{K}$ generated by $\beta, \gamma$, and $\mathbb{F}_p$ is finite, so $\mathbb{K}^\times/(\mathbb{K}^\times)^q$ is cyclic of order $q$. Thus $\Delta^\times/(\Delta^\times)^q = \langle \beta(\Delta^\times)^q \rangle = \langle \gamma(\Delta^\times)^q \rangle$. □

In the case that $\mathbb{F}_a^\times \neq (\mathbb{F}_a^\times)^q$ we fix an element $\alpha_0$ of $\mathbb{F}_a^\times \setminus (\mathbb{F}_a^\times)^q$.

### Lemma 5.
(i) *Let $\Delta = \mathbb{F}(h) \subseteq \mathrm{Mat}(q, \mathbb{F})$ be a field, where $|\pi(h)| = q$. Then $t\Delta t^{-1} = \Delta_\alpha$ for some $\alpha \in \mathbb{F}^\times \setminus (\mathbb{F}^\times)^q$ and $t \in \mathrm{GL}(q, \mathbb{F})$.*

(ii) *If $\Delta \subseteq \mathrm{Mat}(q, \mathbb{F}_a)$ is a field extension of $\mathbb{F}_a 1_q$ of degree $q$ then $t_a \Delta t_a^{-1} = \mathbb{F}_a(l_{\alpha_0})$ for some $t_a \in \mathrm{GL}(q, \mathbb{F}_a)$.*

**Proof.** (i) If $h^q = \beta^q 1_q$, $\beta \in \mathbb{F}^\times$, then $\beta^{-1} h \in \langle \epsilon 1_q \rangle$, contradicting the choice of $h$. Therefore $h^q = \alpha 1_q$, $\alpha \notin (\mathbb{F}^\times)^q$, and by Lemma 2, $tht^{-1} = l_\alpha$ for some $t \in \mathrm{GL}(q, \mathbb{F})$. That is, $t\Delta t^{-1} = \mathbb{F}(tht^{-1}) = \mathbb{F}(l_\alpha)$.
(ii) By Lemma 3, $\Delta = \mathbb{F}_a(h)$ where $h \notin \mathbb{F}_a 1_q$ and $h^q = \beta 1_q \in \mathbb{F}_a^\times 1_q$. Just as in the previous paragraph, we verify that $\beta \notin (\mathbb{F}_a^\times)^q$. Lemma 4 then yields $\beta = \alpha_0^r \gamma^q$ for some $\gamma \in \mathbb{F}_a^\times$ and $r$, $1 \leq r \leq q-1$. By Lemma 2, $tht^{-1} = \gamma l_{\alpha_0}^r$ for some $t \in \mathrm{GL}(q, \mathbb{F})$, so that $t\Delta t^{-1} = t\mathbb{F}_a(h)t^{-1} = \mathbb{F}_a(l_{\alpha_0})$. Since $h, \gamma l_{\alpha_0}^r \in \mathrm{GL}(q, \mathbb{F}_a)$, $t$ may be chosen in $\mathrm{GL}(q, \mathbb{F}_a)$. □

Denote $\mathbb{F}_a(l_{\alpha_0})^\times$ by $D_a$. Certainly $D_a$ is periodic, and irreducible.

### Lemma 6.
$\det(D_a) = \mathbb{F}_a^\times$.

**Proof.** For $q > 2$, or $q = 2$ and $-1 \in (\mathbb{F}_a^\times)^2$, it is clear that $\det(D_a) \nsubseteq (\mathbb{F}_a^\times)^q$, because $l_{\alpha_0} \in D_a$. On the other hand if $q = 2$ and $-1 \notin (\mathbb{F}_a^\times)^2$ then there exists $\begin{pmatrix} \theta & \omega \\ -\omega & \theta \end{pmatrix} \in D_a$, where $\theta, \omega \in \mathbb{F}_p$ and $\theta^2 + \omega^2 = -1$. Now we appeal to Lemma 4. □

### Lemma 7.
*Let $H$ be a periodic abelian irreducible subgroup of $\mathrm{GL}(q, \mathbb{F})$. Then $H$ is conjugate to a subgroup of $D_a$.*

**Proof.** According to Theorem 1, $H$ is conjugate to a subgroup $H_1$ of $\mathrm{GL}(q, \mathbb{F}_a)$. Since $\langle H_1 \rangle_{\mathbb{F}_a}$ is a degree $q$ extension of $\mathbb{F}_a 1_q$, $H_1$ is conjugate to a subgroup of $D_a$ by Lemma 5. □

### Lemma 8.
*Let $\pi(h)$ be an element of order $q$ of $\pi(\Delta_\alpha^\times)$. Then $h = \beta l_\alpha^r$ for some $\beta \in \mathbb{F}^\times$, $1 \leq r \leq q-1$.*

**Proof.** Recall that $\Delta_\alpha = \mathbb{F}(h)$ is a cyclic extension of $\mathbb{F}1_q$. Kummer theory and Lemma 2 tell us that $tht^{-1} = \beta_1 l_\alpha^r$ for some $t \in \mathrm{GL}(q, \mathbb{F})$, $\beta_1 \in \mathbb{F}^\times$, and $1 \leq r \leq q-1$. Therefore $t\mathbb{F}(h)t^{-1} = \mathbb{F}(\beta_1 l_\alpha^r) = \mathbb{F}(l_\alpha) = \Delta_\alpha$; that is, $t \in \mathrm{N}(\Delta_\alpha^\times)$. Write $t = d^m b$, $b \in \Delta_\alpha^\times$. Then $h = \beta_1 t^{-1} l_\alpha^r t = \beta_1 b^{-1} d^{-m} l_\alpha^r d^m b = \beta_1 \epsilon^{-mr} l_\alpha^r$. □

### Lemma 9.

*The group $D_a$ is a maximal periodic subgroup of $\Delta_{\alpha_0}^\times$. If $\alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$ then $\mathbb{F}_a^\times 1_q$ is the unique maximal periodic subgroup of $\Delta_\alpha^\times$.*

**Proof.** Let $P$ be a periodic subgroup of $\Delta_{\alpha_0}^\times$ containing $D_a$. By Lemma 7, $tPt^{-1} \leq D_a$ for some $t \in \mathrm{GL}(q,\mathbb{F})$. Since $\langle P\rangle_\mathbb{F} = \langle D_a\rangle_\mathbb{F} = \Delta_{\alpha_0}$ it follows that $t\Delta_{\alpha_0}t^{-1} = \Delta_{\alpha_0}$, and thus $t = d^m b$ for some $b \in \Delta_{\alpha_0}^\times$ and $m \geq 1$. Since $d^m D_a d^{-m} = D_a$ we get that $tD_at^{-1} = d^m b D_a b^{-1} d^{-m} = D_a$, and then $tPt^{-1} \leq D_a$ implies $P \leq D_a$.
Suppose that $\alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$. If $h \in \Delta_\alpha^\times$ has finite order then $t_1 h t_1^{-1} = h_a \in \mathrm{GL}(q,\mathbb{F}_a)$ for some $t_1 \in \mathrm{GL}(q,\mathbb{F})$. Suppose that $h \notin \mathbb{F}^\times 1_q$. Then $\Delta_\alpha = \mathbb{F}(h)$ and $t_1\Delta_\alpha t_1^{-1} = \mathbb{F}(h_a)$. By Lemma 5, $t_a\mathbb{F}_a(h_a)t_a^{-1} = \mathbb{F}_a(I_{\alpha_0})$ for some $t_a \in \mathrm{GL}(q,\mathbb{F}_a)$. Therefore $t\Delta_\alpha t^{-1} = \mathbb{F}(I_{\alpha_0}) = \Delta_{\alpha_0}$ where $t = t_a t_1$. By Lemma 8, $tI_\alpha t^{-1} = \beta I_{\alpha_0}^r$ for some $\beta \in \mathbb{F}^\times$ and $r$. Hence $\det(I_\alpha) = \det(\beta I_{\alpha_0}^r)$, which gives the contradiction $\alpha \in \mathbb{F}_a^\times(\mathbb{F}^\times)^q$. Thus $h \in \mathbb{F}^\times 1_q$ and so $h \in \mathbb{F}_a^\times 1_q$. $\square$

### Lemma 10.

*If $H$ is a subgroup of $\mathrm{GL}(q,\mathbb{F})$ such that $\det(H) \subseteq \mathbb{F}_a^\times$ and $\pi(H)$ is periodic, then $H$ is periodic.*

**Proof.** Let $h \in H$, $h^n = \beta 1_q \in \mathbb{F}^\times 1_q$. Then $\beta^q = \det(h)^n \in \mathbb{F}_a^\times$, proving $h$ has finite order. $\square$

For a field $\mathbb{K}$, we denote the group of all diagonal matrices in $\mathrm{GL}(q,\mathbb{K})$ by $\mathrm{D}(q,\mathbb{K})$. Let $D_1 = \mathrm{D}(q,\mathbb{F}_a)\mathbb{F}^\times 1_q$. Define irreducible abelian subgroups $D_\alpha$ of $\mathrm{GL}(q,\mathbb{F})$ by

$$D_\alpha = \begin{cases} D_a\mathbb{F}^\times 1_q & \alpha = \alpha_0 \\ \langle I_\alpha, \mathbb{F}^\times 1_q\rangle & \alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q. \end{cases}$$

### Proposition 11.

*The group $\pi(D_1)$ is the unique maximal periodic subgroup of $\pi(\mathrm{D}(q,\mathbb{F}))$, and $\pi(D_\alpha)$ is the unique maximal periodic subgroup of $\pi(\Delta_\alpha^\times)$.*

**Proof.** Let $a = \mathrm{diag}(a_1,\dots,a_q)$ be an element of $\mathrm{D}(q,\mathbb{F})$ such that $a^n \in \mathbb{F}^\times 1_q$. Then $a = \mathrm{diag}(a_1, \epsilon_2 a_1, \dots, \epsilon_q a_1) \in D_1$ where $\epsilon_i \in \mathbb{F}_a^\times$, $\epsilon_i^n = 1$, $i = 2,\dots,q$.
Now we proceed to the second claim. Let $\pi(h)$ be an element of $\pi(\Delta_\alpha^\times)$ of finite order. We show that $h \in D_\alpha$. Assume $|\pi(h)| > 1$ (otherwise $h \in \mathbb{F}^\times 1_q \leq D_\alpha$). Note that if $|\pi(h)| = q$ then $h \in D_\alpha$ by Lemma 8. Set $\det(h) = \gamma$. Then $\det(\gamma^{-1}h^q) = 1$ and, by Lemma 10, $\gamma^{-1}h^q$ is a finite order element of $\Delta_\alpha^\times$. Consequently, if $\alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$ then by Lemma 9, $\gamma^{-1}h^q \in \mathbb{F}_a^\times 1_q$; that is, $|\pi(h)| = q$. Let $\alpha = \alpha_0$, so $\gamma^{-1}h^q \in D_a$. By Lemmas 4 and 6 we have $D_a = \langle c, D_a^q\rangle$, $\det(c) \notin (\mathbb{F}_a^\times)^q$. Hence $\gamma^{-1}h^q = c^m b^q$ for some $b \in D_a$ and $0 \leq m \leq q-1$, which implies $\det(c)^m \in (\mathbb{F}_a^\times)^q$. If $m > 0$ then $\det(c) \in (\mathbb{F}_a^\times)^q$: thus $(hb^{-1})^q = \gamma 1_q$, so $|\pi(hb^{-1})| = q$ or $hb^{-1} \in \mathbb{F}^\times 1_q$. In either situation $hb^{-1} \in D_\alpha$. Since $b \in D_a \leq D_\alpha$, we are done. $\square$

We introduce some more notation:

$$\begin{aligned}
t_{1b} &= Ib, \quad b \in \mathrm{D}(q,\mathbb{F}) \\
t_{2b} &= t_{2b}(\alpha) = db, \quad b \in \Delta_\alpha^\times \\
G_1 &= \mathrm{GL}(q,\mathbb{F}_a)\mathbb{F}^\times 1_q \\
G_2 &= G_2(\alpha, b) = \langle D_\alpha, t_{2b}\rangle, \quad \det(t_{2b}) \notin \langle \alpha, \mathbb{F}_a^\times(\mathbb{F}^\times)^q\rangle \\
G_3 &= G_3(b) = \langle D_1, t_{1b}\rangle, \quad \det(t_{1b}) \notin \det(D_1) = \mathbb{F}_a^\times(\mathbb{F}^\times)^q.
\end{aligned}$$

For $i = 1,2,3$, define $\mathcal{M}_i$ to be the set of subgroups of $\mathrm{GL}(q,\mathbb{F})$ that are conjugate to groups of the form $G_i$, and define $\mathcal{M}_i^* := \{\pi(H) \mid H \in \mathcal{M}_i\}$. Note that for some fields $\mathbb{F}$, $\mathcal{M}_2, \mathcal{M}_3$ are empty; for example this happens if $\mathbb{F}$ is algebraically closed. Denote $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ by $\mathcal{M}$, and $\mathcal{M}_1^* \cup \mathcal{M}_2^* \cup \mathcal{M}_3^*$ by $\mathcal{M}^*$.

**Lemma 12.**
*For all $i = 1, 2, 3$, $G_i$ is irreducible and $\pi(G_i)$ is periodic.*

**Proof.** Obviously $G_1$ and $G_2 \geq D_\alpha$ are irreducible. A $\mathrm{D}(q, \mathbb{F}_a)$-submodule of the underlying space for $\mathrm{GL}(q, \mathbb{F})$ is a direct sum of 1-dimensional submodules. The only such sum invariant under action of $Ib$ is the entire space: in other words $G_3$ is irreducible.
The groups $\pi(G_1)$, $\pi(D_1)$, and $\pi(D_\alpha)$ are periodic. So too is $\pi(G_3)$, for $(Ib)^q = \det(b)1_q$ and $D_1 \trianglelefteq G_3(b)$. Similarly, we observe that $D_\alpha \trianglelefteq G_2(\alpha, b)$ and

$$t_{2b}^q = dbdb\cdots db = dbd^{-1}d^2bd^{-2}\cdots d^{q-1}bd^{-(q-1)}d^qb = \sigma(b)\cdots\sigma^{q-1}(b)b = \det(b)1_q,$$

and the proof is complete. □

**Remark 13.**
$|G_2(\alpha, b)/D_\alpha| = |G_3(b)/D_1| = q$.

**Lemma 14.**
*Let $G$ be an irreducible subgroup of $\mathrm{GL}(q, \mathbb{K})$, where $\mathbb{K}$ is any field. If $G$ is not absolutely irreducible then $G$ is abelian.*

**Proof.** This follows from [14, 1.19, p. 12]. □

**Theorem 15.**
*Each irreducible periodic subgroup of $\mathrm{PGL}(q, \mathbb{F})$ is contained in an element of $\mathcal{M}^*$.*

**Proof.** Let $H$ be an irreducible subgroup of $\mathrm{GL}(q, \mathbb{F})$ containing $\mathbb{F}^\times 1_q$ such that $\pi(H)$ is periodic. Denote the normal subgroup $\{h \in H \mid \det(h) \in \mathbb{F}_a^\times\}$ of $H$ by $B$. By Theorem 1 and Lemma 10 we may assume $B \leq \mathrm{GL}(q, \mathbb{F}_a)$. Suppose that $B$ is absolutely irreducible; then $\langle B\rangle_{\mathbb{F}_a} = \mathrm{Mat}(q, \mathbb{F}_a)$. Each element $g$ of $H$ induces (by conjugation) an automorphism of the simple $\mathbb{F}_a$-algebra $\mathrm{Mat}(q, \mathbb{F}_a)$, so by the Noether-Skolem Theorem $gx_g$ centralises $B$ for some $x_g \in \mathrm{GL}(q, \mathbb{F}_a)$. Since $B$ is absolutely irreducible we have $gx_g \in \mathbb{F}^\times 1_q$, demonstrating that $H \leq G_1$.
Suppose that $B$ is irreducible but not absolutely irreducible. Then $B$ is abelian by Lemma 14, and by Lemma 7 we may further assume $B \leq D_a$. Hence $\langle B\rangle_{\mathbb{F}} = \langle D_a\rangle_{\mathbb{F}} = \Delta_{\alpha_0}$, so $H \leq \mathrm{N}(\Delta_{\alpha_0}^\times)$. By Proposition 11, $H \leq \mathrm{N}(D_{\alpha_0})$ and $H \cap \Delta_{\alpha_0}^\times = H \cap D_{\alpha_0}$. Then $|HD_{\alpha_0}/D_{\alpha_0}| = |H\Delta_{\alpha_0}^\times/\Delta_{\alpha_0}^\times|$ divides $|\mathrm{N}(\Delta_{\alpha_0}^\times)/\Delta_{\alpha_0}^\times| = q$, and thus $H \leq \langle D_{\alpha_0}, t_{2b}\rangle$ where $t_{2b} = db$, $b \in \Delta_{\alpha_0}^\times$. If $H \not\leq D_{\alpha_0}$ and $\det(t_{2b}) \in \mathbb{F}_a^\times(\mathbb{F}^\times)^q$ then $H = \langle t_{2b}, H \cap D_{\alpha_0}\rangle$ and $\det(ct_{2b}) \in \mathbb{F}_a^\times$ for some $c \in \mathbb{F}^\times$; that is $dbc \in B \leq \Delta_{\alpha_0}^\times$, and since $bc \in \Delta_{\alpha_0}^\times$ it follows that $d \in \Delta_{\alpha_0}^\times$. This contradiction forces $H \leq G_1$ or $H \leq G_2(\alpha_0, b)$.
Suppose that $H$ is primitive and $B$ is reducible, so that $B = \mathbb{F}_a^\times 1_q$ by Clifford's Theorem. As $[H, H] \leq B$, $H$ is abelian or class 2 nilpotent. For the moment let $H$ be nonabelian. The irreducible maximal metabelian subgroups of $\mathrm{GL}(n, \mathbb{K})$ for any field $\mathbb{K}$ are classified in [9, Theorem 1]. By that result, $H$ is conjugate to a subgroup of $H_1 := \langle Ic, db, \mathbb{F}^\times 1_q\rangle$ where $c = \mathrm{diag}(1_{q-1}, c_1)$, $c_1 \in \mathbb{F}^\times$, and $b \in \mathrm{GL}(q, \mathbb{F})$ commutes with $Ic$. If $c_1 = c_2c_3^q$, $c_2 \in \mathbb{F}_a^\times$, $c_3 \in \mathbb{F}^\times$, then $\det(c_3^{-1}Ic) = \pm c_2$; but $B$ is scalar. Thus $Ic = I_\alpha$, $\alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$, and $b \in \mathrm{C}(I_\alpha) = \Delta_\alpha^\times$. That is, $H_1 = \langle D_\alpha, t_{2b}\rangle$. If $\det(t_{2b}) = \nu^q\alpha^s\gamma$ for some $\nu \in \mathbb{F}^\times$, $\gamma \in \mathbb{F}_a^\times$, and integer $s$, then $\nu^{-1}I_\alpha^{-s}t_{2b} \in B$, which yields the absurdity $d \in \Delta_\alpha^\times$. We have verified that $H \leq G_2(\alpha, b)$, $H$ nonabelian. If $H$ is abelian then $\langle H\rangle_{\mathbb{F}} = \mathbb{F}(h)$ is a simple extension of $\mathbb{F}1_q$. Let $\det(h) = \gamma$, $|\pi(h)| = r > 1$, and $\beta$ be an element of $\mathbb{F}^\times$ such that $h^r = \beta 1_q$. If $\beta^m = \beta_1^r$ for some $\beta_1 \in \mathbb{F}^\times$, $m \geq 1$, then $h^{mr} = \beta^m 1_q$ and $\gamma^{mr} = \beta^{mq} = \beta_1^{rq}$. That is, $\gamma^m = \delta\beta_1^q$ where $\delta \in \mathbb{F}_a^\times$, $\delta^r = 1$. Thus $\det(\beta_1^{-1}h^m) \in \mathbb{F}_a^\times$, so $\beta_1^{-1}h^m \in B$ and $h^m \in \mathbb{F}^\times 1_q$. Then since $h \notin \mathbb{F}^\times 1_q$, $m = r$ if $m$ divides $r$. Now $\beta^q = \gamma^r \in (\mathbb{F}^\times)^r$, so that $\beta \in (\mathbb{F}^\times)^r$ if $q$ does not divide $r$; however, we then infer $r = 1$ from the preceding (with $m = 1$). Hence $q$ divides $r$ and, again by the preceding (with $m = q$), $r = q$. Thus $|\pi(h)| = q$ and by Lemma 5 and Proposition 11, $H$ is contained in an element of $\mathcal{M}_2$.
Now let $H$ be imprimitive. In prime degree $q$, this means $H$ is monomial: $H$ is a subgroup of the full monomial group $\mathrm{D}(q, \mathbb{F}) \rtimes \mathrm{Sym}(q)$, up to conjugacy. Since $H$ normalises $\mathrm{D}(q, \mathbb{F})$, it follows from Proposition 11 that

$H \cap \mathrm{D}(q, \mathbb{F}) = H \cap D_1$. Since $H$ is irreducible, $HD_1/D_1$ is isomorphic to a transitive subgroup of $\mathrm{Sym}(q)$. If $B \not\leq D_1$ then $BD_1/D_1$, as a nontrivial normal subgroup of the transitive prime degree permutation group $HD_1/D_1$, is transitive. Therefore $BD_1$ is irreducible. By Lemma 14, $BD_1$ is absolutely irreducible. As shown at the beginning of the proof, $H\mathrm{D}(q, \mathbb{F}_a)$ is then conjugate to a subgroup of $G_1$. Hence we may take $B \leq \mathrm{D}(q, \mathbb{F}_a)$.
Let $h \in H \setminus H \cap D_1$, and set $\det(h) = \eta$. Then $\eta^{-1}h^q \in B$, implying $h^q \in D_1$. Thus $H/H \cap D_1$ is isomorphic to a $q$-subgroup of $\mathrm{Sym}(q)$, and so $|H/H \cap D_1| = q$. Moreover $H$ is conjugate to $\langle H \cap D_1, t_{1b} \rangle$ for some $b \in \mathrm{D}(q, \mathbb{F})$. Since $B$ is diagonal but $I$ is not, $\det(t_{1b}) \notin \mathbb{F}_a^\times (\mathbb{F}^\times)^q$. Hence $H$ is contained in an element of $\mathcal{M}_3$. □

### Lemma 16.

*For $H \in \mathcal{M}_i$ and $i = 2, 3$, $|\det(H)/(\mathbb{F}^\times)^q| = q^2$. If $\mathbb{F}_a^\times (\mathbb{F}^\times)^q \subseteq \det(G_2(\alpha, b))$ then $\alpha = \alpha_0$.*

**Proof.** Let $H = \langle D_\alpha, t_{2b} \rangle$, $\det(t_{2b}) \notin \langle \alpha, \mathbb{F}_a^\times (\mathbb{F}^\times)^q \rangle$. If $\alpha = \alpha_0$ then $\det(D_\alpha) = \mathbb{F}_a^\times (\mathbb{F}^\times)^q = \langle \alpha_0, (\mathbb{F}^\times)^q \rangle$ by Lemma 6, and if $\alpha \notin \mathbb{F}_a^\times (\mathbb{F}^\times)^q$ then $\det(D_\alpha) = \langle (-1)^{q-1}\alpha, (\mathbb{F}^\times)^q \rangle$; in both cases $\det(H)/(\mathbb{F}^\times)^q$ is an elementary abelian $q$-group of rank 2. Moreover if $\mathbb{F}_a^\times (\mathbb{F}^\times)^q \subseteq \det(G_2(\alpha, b))$ then $\alpha = \alpha_0$, for otherwise $\det(t_{2b}) \in \langle \alpha_0, (-1)^{q-1}\alpha, (\mathbb{F}^\times)^q \rangle \leq \langle \alpha, \mathbb{F}_a^\times (\mathbb{F}^\times)^q \rangle$. The rest of the proof is left as an exercise. □

We now strengthen Theorem 15, thereby showing that each irreducible periodic subgroup of $\mathrm{PGL}(q, \mathbb{F})$ is contained in an element of $\mathcal{M}^*$. This affords a complete and explicit description of the maximal irreducible periodic subgroups of $\mathrm{PGL}(q, \mathbb{F})$ up to conjugacy.

### Theorem 17.

*Each group in $\mathcal{M}^*$ is an irreducible maximal periodic subgroup of* $\mathrm{PGL}(q, \mathbb{F})$.

**Proof.** By Lemma 12, we only have to prove the maximality assertions. Let $H^* := \pi(H) \in \mathcal{M}^*$, and let $L^* := \pi(L)$ be a periodic subgroup of $\mathrm{PGL}(q, \mathbb{F})$, $H \leq L \leq \mathrm{GL}(q, \mathbb{F})$. By Theorem 15, $tHt^{-1} \leq tLt^{-1} \leq G_i$ for some $i$ and $t \in \mathrm{GL}(q, \mathbb{F})$.
Let $H = G_1$. Then $H$ is primitive, and since groups in $\mathcal{M}_3$ are monomial, $i \neq 3$. Suppose that $tLt^{-1} \leq G_2(\alpha, b)$, so $\det(H) = \mathbb{F}_a^\times (\mathbb{F}^\times)^q \subseteq \det(G_2(\alpha, b))$. By Lemma 16, $G_2(\alpha, b) = \langle D_{\alpha_0}, t_{2b} \rangle$, $\det(t_{2b}) \notin \mathbb{F}_a^\times (\mathbb{F}^\times)^q = \det(D_{\alpha_0})$. Thus $tHt^{-1} \leq D_{\alpha_0}$. However $H$ is certainly not abelian. Therefore $i = 1$; that is, $t \in \mathrm{N}(G_1) = G_1$, and so $H = L = G_1$.
Let $H = G_2(\alpha, b)$. Since $\det(H) \not\subseteq \mathbb{F}_a^\times (\mathbb{F}^\times)^q$, $i$ is not 1. Suppose that $tLt^{-1} \leq G_3(b_1)$. By Lemma 16, $\det(H) = \det(G_3(b_1)) \supseteq \mathbb{F}_a^\times (\mathbb{F}^\times)^q$ and $\alpha = \alpha_0$, so $H = \langle D_{\alpha_0}, t_{2b} \rangle$, $\det(t_{2b}) \notin \det(D_{\alpha_0}) = \mathbb{F}_a^\times (\mathbb{F}^\times)^q$. Since $\det(t_{1b_1}) \notin \det(D_1) = \mathbb{F}_a^\times (\mathbb{F}^\times)^q$, it follows that $tD_{\alpha_0}t^{-1} \leq D_1$. However $tD_{\alpha_0}t^{-1}$ is irreducible, hence not a subgroup of $\mathrm{D}(q, \mathbb{F})$. This contradiction leaves us to consider that $tLt^{-1} \leq G_2(\alpha_1, b_1)$. By Lemma 16, $\det(G_2(\alpha, b)) = \det(G_2(\alpha_1, b_1))$, and $\alpha = \alpha_0$ if and only if $\alpha_1 = \alpha_0$. Suppose that $\alpha = \alpha_0$, so $\det(t_{2b})$, $\det(t_{2b_1}) \notin \det(D_{\alpha_0}) = \mathbb{F}_a^\times (\mathbb{F}^\times)^q$, and $t \in \mathrm{N}(D_{\alpha_0})$. Hence

$$tHt^{-1} = t\langle D_{\alpha_0}, t_{2b} \rangle t^{-1} = \langle D_{\alpha_0}, tt_{2b}t^{-1} \rangle \leq tLt^{-1} \leq \langle D_{\alpha_0}, t_{2b_1} \rangle.$$

Since $|G_2(\alpha_0, b)/D_{\alpha_0}| = |G_2(\alpha_0, b_1)/D_{\alpha_0}| = q$, we get $H = L$. Similarly, if $\alpha \neq \alpha_0$ then $|G_2(\alpha, b)/\mathbb{F}^\times 1_q| = |G_2(\alpha, b_1)/\mathbb{F}^\times 1_q| = q^2$, and as a consequence $H = L$.
Let $H = G_3(b)$. We immediately rule out $i = 1$ because $\det(G_3(b)) \not\subseteq \mathbb{F}_a^\times (\mathbb{F}^\times)^q = \det(G_1)$. Suppose that $tLt^{-1} \leq G_2(\alpha, b_1)$. By Lemma 16, $\alpha = \alpha_0$. Since $\det(t_{2b_1}) \notin \mathbb{F}_a^\times (\mathbb{F}^\times)^q$ we have $tD_1t^{-1} \leq D_{\alpha_0}$, so that the element $t^{-1}I_{\alpha_0}t$ of $\mathrm{GL}(q, \mathbb{F})$ centralising $\mathrm{D}(q, \mathbb{F}_a)$ must itself be diagonal. However $\langle t^{-1}I_{\alpha_0}t \rangle$ is irreducible. Finally, suppose that $tLt^{-1} \leq G_3(b_1)$. Since then $t$ normalises $D_1$, and $|G_3(b)/D_1| = |G_3(b_1)/D_1| = q$, we get $tHt^{-1} = tLt^{-1}$ as required. □

### Remark 18.

By Theorem 15, if $\mathbb{F}_a$ is finite then every irreducible periodic subgroup of $\mathrm{PGL}(q, \mathbb{F})$ is finite. Even if $\mathbb{F}_a$ is infinite, $\mathrm{PGL}(q, \mathbb{F})$ can have finite periodic subgroups: $\pi(H)$, where $H = G_2(\alpha, b)$, $\alpha \neq \alpha_0$, is finite. In fact $\pi(H)$ is a Sylow $q$-subgroup of $\mathrm{PGL}(q, \mathbb{F})$ of order $q^2$.

**Corollary 19.**
*If $(\mathbb{F}^\times)^q = \mathbb{F}^\times$ (e.g. if $\mathbb{F}$ is algebraically closed) then every irreducible maximal periodic subgroup of $\mathrm{PGL}(q,\mathbb{F})$ is conjugate to $\pi(G_1) \cong \mathrm{PGL}(q,\mathbb{F}_a)$.*

Next, we investigate conjugacy between groups in $\mathcal{M}^*$. Of course, it is sufficient to determine when groups in $\mathcal{M}$ are $\mathrm{GL}(q,\mathbb{F})$-conjugate.

**Proposition 20.**
*Groups in different lists $\mathcal{M}_i$, $i = 1,2,3$, are not conjugate.*

**Proof.** This has already been established in the proof of Theorem 17 (take $L = H$ there). $\square$

Groups in $\mathcal{M}_1$ are pairwise conjugate, by definition. Several auxiliary results are needed to obtain criteria for determining conjugacy between groups in the same list $\mathcal{M}_2$ or $\mathcal{M}_3$.

**Lemma 21.**
*Suppose that $H \in \mathcal{M}_2$, $\det(H) \not\supseteq \mathbb{F}_a^\times(\mathbb{F}^\times)^q$, and $(-1)^{q-1}\alpha \in \det(H)$, $\alpha \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$. Then $H$ is conjugate to $G_2(\alpha,b) \in \mathcal{M}_2$.*

**Proof.** Without loss of generality let $H = G_2(\alpha_1,b_1)$, $\alpha_1 \notin \mathbb{F}_a^\times(\mathbb{F}^\times)^q$.
Suppose that $(-1)^{q-1}\alpha \in \det(D_{\alpha_1}) = \langle(-1)^{q-1}\alpha_1,(\mathbb{F}^\times)^q\rangle$. That is, $\alpha = \alpha_1^r c^q$ for some $c \in \mathbb{F}^\times$ and $1 \le r \le q-1$. By Lemma 2, $tD_{\alpha_1}t^{-1} = D_\alpha$ for some $t \in \mathrm{GL}(q,\mathbb{F})$, and then $tHt^{-1} = \langle D_\alpha, g\rangle$, where $g = tt_{2b_1}t^{-1}$. Since $D_{\alpha_1} \trianglelefteq H$, so $D_\alpha \trianglelefteq tHt^{-1}$.
Suppose that $(-1)^{q-1}\alpha \notin \det(D_{\alpha_1})$. In this case $\det(t_{2b_1}a) = (-1)^{q-1}\alpha^r$ for some $a \in D_{\alpha_1}$ and $1 \le r \le q-1$, whereby we may assume $\det(t_{2b_1}) = (-1)^{q-1}\alpha^r$, and thus $\det(b_1) = \alpha^r$. Recall from the proof of Lemma 12 that $t_{2b_1}^q = \det(b_1)1_q$. By Lemma 2, $tt_{2b_1}t^{-1} = I_\alpha^r$ for some $t \in \mathrm{GL}(q,\mathbb{F})$. Therefore $tH't^{-1} = D_\alpha$ where $H' = \langle t_{2b_1}, \mathbb{F}^\times 1_q\rangle$. Also $H = \langle H', I_{\alpha_1}\rangle$ and $tHt^{-1} = \langle D_\alpha, tI_{\alpha_1}t^{-1}\rangle$. Note that $D_\alpha \trianglelefteq tHt^{-1}$. Indeed, since $I_{\alpha_1}d = \epsilon^{-1}dI_{\alpha_1}$ and $I_{\alpha_1}, b_1 \in \Delta_{\alpha_1}^\times$ commute, $I_{\alpha_1}t_{2b_1}I_{\alpha_1}^{-1} = I_{\alpha_1}db_1I_{\alpha_1}^{-1} = \epsilon^{-1}db_1 = \epsilon^{-1}t_{2b_1}$. Thus $H' \trianglelefteq H$ and $D_\alpha \trianglelefteq tHt^{-1}$.
We have shown that $H$ is conjugate to a group $\langle D_\alpha, g\rangle$ where $g \in \mathrm{N}(D_\alpha)$, so that $g = d^m g_1$ for some $g_1 \in \Delta_\alpha^\times$ and $1 \le m \le q-1$ ($H$ is nonabelian). Thus $\langle D_\alpha, g\rangle = \langle D_\alpha, t_{2b}\rangle$, $b \in \Delta_\alpha^\times$. If $\det(t_{2b}) \in \langle\alpha, \mathbb{F}_a^\times(\mathbb{F}^\times)^q\rangle$ then $\det(H) \subseteq \langle\alpha, \mathbb{F}_a^\times(\mathbb{F}^\times)^q\rangle$ and by Lemma 16, $\det(H) = \langle\alpha, \mathbb{F}_a^\times(\mathbb{F}^\times)^q\rangle$. Therefore $H$ is conjugate to $G_2(\alpha,b) \in \mathcal{M}_2$ as claimed. $\square$

**Lemma 22.**
*Define $t_a = Ia$ and $t_b = Ib$, where $a = \mathrm{diag}(a_2,\dots,a_q,a_1), b = \mathrm{diag}(b_2,\dots,b_q,b_1)$. Let $g \in \mathrm{D}(q,\mathbb{F})$. Then $gt_ag^{-1} = t_b$ if and only if $\det(t_a) = \det(t_b)$ and $g = \beta\,\mathrm{diag}(g_2,\dots,g_q,1)$, where $g_i = a_i \cdots a_q(b_i \cdots b_q)^{-1}$, $2 \le i \le q$, and $\beta \in \mathbb{F}^\times$.*

**Proof.** Let $g = \mathrm{diag}(c_1,\dots,c_q)$. Easy calculations show that $gt_ag^{-1} = t_b$ if and only if $a_1 \cdots a_q = b_1 \cdots b_q$ and $c_i = a_{i+1}b_{i+1}^{-1}c_{i+1}$, $1 \le i \le q-1$. Then by recursion $c_i = a_{i+1}\cdots a_q(b_{i+1}\cdots b_q)^{-1}c_q$. Hence $g = c_q\,\mathrm{diag}(g_2,\dots,g_q,1)$. $\square$

**Lemma 23.**
*Let $g, b_i \in \Delta_\alpha^\times$, $i = 1,2$, and put $t_i = db_i$. Then $gt_1g^{-1} = t_2$ if and only if $\det(t_1) = \det(t_2)$ and $t_2 = t_1a$, where $a = \sigma^{-1}(g)g^{-1}$.*

**Proof.** Suppose that $gt_1g^{-1} = t_2$. Then $t_1a = db_1d^{-1}gdg^{-1} = dd^{-1}gdb_1g^{-1} = gt_1g^{-1} = t_2$. Conversely, if $t_2 = t_1a$ then $gt_1g^{-1} = gt_2a^{-1}g^{-1} = gt_2\sigma^{-1}(g^{-1})gg^{-1} = gdb_2d^{-1}g^{-1}d = gg^{-1}db_2d^{-1}d = db_2 = t_2$. $\square$

**Proposition 24.**
(i) *$H_1, H_2 \in \mathcal{M}_3$ are conjugate if and only if $\det(H_1) = \det(H_2)$.*

(ii) $H_i = \langle D_{\alpha_i}, t_{2b_i}\rangle \in \mathcal{M}_2$, $i = 1, 2$, *are conjugate if and only if* $\det(H_1) = \det(H_2)$ *and* $\det(b_1) = \det(b_2c)$ *for some* $c \in D_\alpha$.

***Proof.*** (i) Let $H_i = \langle D_1, t_{1b_i}\rangle \in \mathcal{M}_3$, $i = 1, 2$, and suppose that $\det(H_1) = \det(H_2)$. Since $\det(t_{1b_i}) \notin \det(D_1)$, we have $\det(t_{1b_1}) = \det((t_{1b_2})^r c)$ for some $c \in D_1$, $1 \le r \le q-1$. There exist a permutation matrix $x$ and diagonal matrix $b$ such that $x(t_{1b_2})^r c x^{-1} = t_{1b}$. Then by Lemma 22, $H_1$ and $H_2$ are conjugate (by a monomial matrix).
(ii) By virtue of Lemmas 16 and 21 we may assume $\alpha_1 = \alpha_2 = \alpha$. Suppose that $\det(H_1) = \det(H_2)$ and $\det(b_1) = \det(b_2')$, $b_2' = b_2c$, $c \in D_\alpha$. By Hilbert's Theorem 90, $b_2'b_1^{-1} = \sigma^{-1}(g)g^{-1}$ for some $g \in \Delta_\alpha^\times$. Hence $gt_{2b_1}g^{-1} = t_{2b_2'} = t_{2b_2}c$ by Lemma 23, and since $gD_\alpha g^{-1} = D_\alpha$, so $gH_1g^{-1} = H_2$.
Now suppose that $gH_1g^{-1} = H_2$, $g \in \mathrm{GL}(q,\mathbb{F})$. If $g \notin \mathrm{N}(D_\alpha)$ then $t_{2b_1} \in D_\alpha(g^{-1}D_\alpha g)$, implying $\det(t_{2b_1}) \in \det(D_\alpha)$, in violation of Lemma 16. Thus $g \in \mathrm{N}(D_\alpha) = \langle \Delta_\alpha^\times, d\rangle$, and then $t_{2b_1'}a = t_{2b_2}^r c$ for a conjugate $b_1'$ of $b_1$ in $\Delta_\alpha^\times$, $a \in \Delta_\alpha^\times$ such that $\det(a) = 1$, $c \in D_\alpha$, and $1 \le r \le q-1$. Obviously $r = 1$, so $\det(b_1) = \det(b_2c)$. □

We round out the paper with Proposition 26 below, which is another interesting fact about conjugacy between irreducible periodic subgroups of $\mathrm{PGL}(q,\mathbb{F})$.

***Lemma 25.***
$\mathrm{N}(\mathrm{D}(q,\mathbb{F}_a)) = \mathrm{N}(\mathrm{D}(q,\mathbb{F}))$ *is the full monomial subgroup of* $\mathrm{GL}(q,\mathbb{F})$.

***Proof.*** This follows easily from Clifford's theorem. □

***Proposition 26.***
*Groups in* $\mathcal{M}^*$ *are self-normalising in* $\mathrm{PGL}(q,\mathbb{F})$.

***Proof.*** We show that $N := \mathrm{N}(H) = H$ for each $H \in \mathcal{M}$. If $H = G_1$ then $N = H$ is clear. Let $H = \langle D_\alpha, t_{2b}\rangle \in \mathcal{M}_2$ and $g \in N$. Then $g \in \mathrm{N}(D_\alpha) = \mathrm{N}(\Delta_\alpha^\times)$ and since $t_{2b} \in N \setminus N_1$ where $N_1 = N \cap \Delta_\alpha^\times$, we get that $N = \langle N_1, t_{2b}\rangle$. If $g \in N_1$ then $gt_{2b}g^{-1} = t_{2b}\sigma^{-1}(g)g^{-1} \in H$, so $c = \sigma^{-1}(g)g^{-1} \in H$. For some $n \ge 1$, $c^n$ is a scalar $\beta 1_q$ of order $q$ ($\det(c) = 1$). Thus $\sigma^{-1}(g^{qn}) = g^{qn}$. That is, $g^{qn}$ is scalar, and so by Proposition 11, $g \in D_\alpha$. Hence $N_1 = D_\alpha$, and $N = H$.
Let $H = \langle D_1, t_{1b}\rangle \in \mathcal{M}_3$. Since $\det(t_{1b}) \notin \det(D_1)$, so $gD_1g^{-1} = D_1$ for all $g \in N$, and $gt_{1b}g^{-1}D_1 = t_{1b}^r D_1$ for some $r$, $1 \le r \le q-1$. We see that $r$ must be 1; otherwise $\det(t_{1b}) \in \det(D_1)$. Thus by Lemma 25, and because the centraliser in $\mathrm{Sym}(q)$ of the Sylow $q$-subgroup $\langle I\rangle$ is $\langle I\rangle$ itself, $g \in \langle \mathrm{D}(q,\mathbb{F}), I\rangle$. Since $t_{1b} \in N$ but $t_{1b} \notin N_2 := N \cap \mathrm{D}(q,\mathbb{F})$, we have $N = \langle N_2, t_{1b}\rangle$. Let $g \in N_2$. Then $gt_{1b}g^{-1} = t_{1b}c$ for some $c \in D_1$. By Lemma 10, $c$ is periodic, say $c = \mathrm{diag}(\epsilon_2, \dots \epsilon_q, \epsilon_1) \in \mathrm{D}(q,\mathbb{F}_a)$. By Lemma 22, it follows that $g = \beta\mathrm{diag}(g_2, \dots, g_q, g_1)$, $\beta \in \mathbb{F}^\times$, $g_i = (\epsilon_i \cdots \epsilon_q)^{-1}$. Therefore $g \in D_1$, $N_2 = D_1$, and once more $N = H$. □

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