SMALL-AMPLITUDE INHOMOGENEOUS PLANE WAVES IN A DEFORMED MOONEY–RIVLIN MATERIAL

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Summary

The propagation of small-amplitude inhomogeneous plane waves in an isotropic homogeneous incompressible Mooney–Rivlin material is considered when the material is maintained in a state of finite static homogeneous deformation. Disturbances of complex exponential type are sought and all propagating inhomogeneous solutions to the equations of motion are given, as well as the conditions for linear, elliptical, or circular polarization. It is seen that a great variety of solutions arises. These include some original solutions, such as circularly polarized plane waves which propagate with an arbitrary complex scalar slowness, or linearly polarized waves for which the direction of propagation is not necessarily orthogonal to the direction of attenuation. Throughout the paper, geometrical interpretations and explicit examples are presented.

1. Introduction

The theoretical study of elastic plane waves has generated a large literature in the area of finite elasticity. One type of plane waves which is of practical interest is that of motions propagating in a finitely and homogeneously deformed elastic material (**1**), because the predeformation can be used to model, at least locally, the anisotropy of many mechanical and geological structures. A review of these topics can be found in a textbook by Ieşan (**2**). Once it is homogeneously deformed, an elastic body presents three privileged orthogonal directions, namely those of the principal axes of the static deformation. Waves propagating in one of these directions are called 'principal waves' (**3**) and their properties are well known and relatively easy to establish, because the principal directions offer a natural rectangular Cartesian coordinate system. Homogeneous plane waves propagating in an incompressible elastic material are necessarily transverse (the direction of propagation is orthogonal to the plane of polarization). Therefore, for such waves propagating in a non-principal direction, another rectangular Cartesian coordinate system proves to be useful, that formed by the direction of propagation and by two orthogonal directions in the plane of polarization. Currie and Hayes (**4**) proved that the Mooney–Rivlin form for the strain energy density (which is used to model the mechanical behaviour of rubber (**5**,**6**)), is the most general one for which homogeneous plane waves, be they of finite or of small amplitude, may propagate in *any* direction for an arbitrary finite static homogeneous predeformation. Later, Boulanger and Hayes (**7**, **8**) studied in great detail finite-amplitude homogeneous plane waves in a deformed Mooney–Rivlin material.

Homogeneous plane waves are such that the planes of constant phase (orthogonal to the direction of propagation) are parallel to the planes of constant amplitude (orthogonal to the direction of eventual attenuation). However, for a variety of physical problems, an attenuation of the amplitude occurs in a direction distinct from the direction of propagation. In those cases, a combination of

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'inhomogeneous' plane waves is introduced, usually in the form $e^{-\omega \mathbf{S}'' \cdot \mathbf{x}} \cos \omega(\mathbf{S}' \cdot \mathbf{x} - t)\mathbf{a}$, where ω is the frequency, and $\mathbf{S}'$, $\mathbf{S}''$, $\mathbf{a}$ are vectors. This procedure was successfully applied to various interfacial problems, such as reflected and refracted waves, Rayleigh waves, Love waves, Stoneley waves, Scholte waves, etc. When the directions of propagation (that of the vector $\mathbf{S}'$), of exponential attenuation (that of the vector $\mathbf{S}''$), and of polarization (that of the vector $\mathbf{a}$) are orthogonal with respect to one another, they form the basis for a rectangular Cartesian coordinate system in which the incremental equations of motion can be written and solved for small-amplitude (**9**, **10**) as well as for finite-amplitude (**11**) inhomogeneous waves in a deformed Mooney–Rivlin material. When these directions are not orthogonal, the equations of motion become much harder to solve. This is where the algebra of complex vectors, also known as 'bivectors', can play a very useful role. The use of bivectors in order to describe inhomogeneous plane waves is made clear in a textbook by Boulanger and Hayes (**12**). Indeed bivectors, that is, vectors with a real part and an imaginary part, can be used to describe the polarization of a wave through the 'polarization bivector' $\mathbf{A} = \mathbf{A}' + i\mathbf{A}''$, as well as its propagation and attenuation through the 'slowness bivector' $\mathbf{S} = \mathbf{S}' + i\mathbf{S}''$. Hence an inhomogeneous plane wave is modelled as being proportional to the real part of the expression $\mathbf{A}e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}$.

Previously, the propagation of small-amplitude *inhomogeneous* plane waves in a Mooney–Rivlin material subjected to a finite static *biaxial* homogeneous deformation has been considered first by Belward (**13**), and later by Boulanger and Hayes (**14**), using bivectors. In this paper, we consider the propagation of infinitesimal inhomogeneous plane waves in a Mooney–Rivlin material which is maintained in a state of arbitrary *triaxial* finite static homogeneous deformation. Most results obtained here are a generalization and an extension to the case of inhomogeneous waves of results established by Boulanger and Hayes (**7**) for the propagation of finite-amplitude *homogeneous* plane waves in a homogeneously deformed Mooney–Rivlin material. However, this extension is only possible when the amplitude of the wave is considered small enough to allow linearization. This restriction is due to the fact that finite-amplitude inhomogeneous plane waves can propagate in an incompressible elastic material only when they are linearly polarized (**15**). For infinitesimal inhomogeneous plane waves, no such restriction applies and elliptical polarization is possible. The purpose of this paper is to find all inhomogeneous small-amplitude plane waves of complex exponential type travelling in a deformed Mooney–Rivlin material, and to establish the conditions for linear, elliptical, and circular polarization. Within this context, a much greater number of solutions is found for inhomogeneous waves than for homogeneous waves. For instance, elliptical polarization is possible for homogeneous waves only in two special directions (the 'acoustic axes' (**7**)), whereas it is possible in other directions for inhomogeneous waves. Also, for certain inhomogeneous waves, the 'complex scalar slowness', which is the counterpart of the inverse of the speed for homogeneous waves, may be arbitrarily prescribed. In general, for a given orientation of the plane containing the directions of propagation and of attenuation, there is an infinity of inhomogeneous wave solutions.

The paper is organized as follows. First (section 2) we recall the basic equations describing the Mooney–Rivlin material. Then (section 3) we write the equations governing the motion of a small-amplitude disturbance in a homogeneously deformed Mooney–Rivlin material. The incremental equations of motion and the incompressibility constraint are given. In order to solve these equations, we seek solutions of complex exponential type, which are presented in section 4. The slowness bivector $\mathbf{S}$ is introduced; when the real and the imaginary parts of $\mathbf{S}$ are not parallel, the wave is inhomogeneous, because the directions of propagation and of attenuation do not coincide. The amplitude bivector $\mathbf{A}$ is also introduced; depending on whether or not the real and the imaginary parts of **A** are parallel, the wave is linearly or elliptically polarized, respectively. The special case of circular polarization corresponds to the 'isotropy' of **A**, that is $\mathbf{A} \cdot \mathbf{A} = 0$ (**16**). Using these quantities, we then derive the incremental equations of motion for inhomogeneous plane waves of complex exponential type propagating in the deformed Mooney–Rivlin material.

Next (sections 5 and 6) the propagation of such disturbances is investigated. Different sub-cases arise, depending on whether or not the slowness complex vector **S** is isotropic, and on whether the waves are circularly, elliptically, or linearly polarized. For each case, the general solution is provided (amplitude and slowness bivectors, incremental pressure, wave speed) as well as various explicit solutions to the incremental equations of motion.

2. Basic equations

The Mooney–Rivlin material is a homogeneous isotropic hyperelastic solid, for which the strain energy Σ per unit volume is given by (**5**)

$$2\Sigma = C(I-3) + D(II-3), \tag{2.1}$$

where C, D are material constants, and I, II are the first and second invariants of the left Cauchy–Green strain tensor $\mathbb{B}$. In order to satisfy the strong ellipticity condition, the constants C, D are such that (**8, 17**) either $C > 0$, $D \geqslant 0$ or $C \geqslant 0$, $D > 0$. When $D = 0$, the material is said to be 'neo-Hookean', but this possibility is not considered in this paper.

The strain tensor $\mathbb{B}$ is related to the deformation gradient $\mathbb{F}$ through $\mathbb{B} = \mathbb{F}\mathbb{F}^{\mathrm{T}}$, and its first two invariants I and II are defined by $I = \operatorname{tr}\mathbb{B}$ and $2II = (\operatorname{tr}\mathbb{B})^2 - \operatorname{tr}(\mathbb{B}^2)$.

Because the Mooney–Rivlin material is incompressible, we have at all times,

$$\det\mathbb{F} = (\det\mathbb{B})^{1/2} = 1. \tag{2.2}$$

Finally, the Mooney–Rivlin constitutive equation is derived from (2.1) and (2.2) as (**7**)

$$\boldsymbol{\sigma} = -(p - DII)\mathbf{1} + C\mathbb{B} - D\mathbb{B}^{-1}, \tag{2.3}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress and $-p\mathbf{1}$ a hydrostatic pressure, to be determined from the equations of motion, and initial and boundary conditions.

Throughout the paper, we assume that no body forces are applied, so that the equations of motion reduce to

$$\operatorname{div}\boldsymbol{\sigma} = \rho(\partial^2\mathbf{x}/\partial t^2). \tag{2.4}$$

Here, $\mathbf{x}$ is the position at time t of a point of the deformed body which was at $\mathbf{X}$ in the undeformed state, and ρ is the constant mass density.

3. Small motions superposed on a large homogeneous deformation

In this section we give the incremental equations of motion corresponding to the superposition of a small-amplitude motion upon a large homogeneous strain. First, we assume that the Mooney–Rivlin material is subjected to a finite pure homogeneous deformation, for which a point initially at $\mathbf{X}$ in the rectangular Cartesian coordinate system $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ moves to a point at $\mathbf{x}$ in the same system, with extension ratios λ_α ($\alpha = 1, 2, 3$) and principal axes of deformation along the axes of the coordinate system, so that

$$x_\alpha = \lambda_\alpha X_\alpha \quad (\alpha = 1, 2, 3;\ \text{no sum}). \tag{3.1}$$

The extension ratios are assumed to be distinct, constant, and without loss of generality, to be ordered as $\lambda_1 > \lambda_2 > \lambda_3$.

The deformation gradient $\mathbb{F}$ and strain tensor $\mathbb{B}$ corresponding to this finite static deformation are constant and given by $\mathbb{F} = \text{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ and $\mathbb{B} = \text{diag}\,(\lambda_1^2, \lambda_2^2, \lambda_3^2)$. Then the tensor $\mathbb{T}$, defined by (**7**, **9**)

$$\begin{aligned}
&\mathbb{T}_{\alpha\beta} = 0, \quad \alpha \neq \beta, \\
&\mathbb{T}_{\alpha\alpha} = -p + D(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}) + C\lambda_\alpha^{-2} - D\lambda_\alpha^{-2} \quad (\alpha = 1, 2, 3;\ \text{no sum}),
\end{aligned} \tag{3.2}$$

where p is constant, is the constant Cauchy stress required to sustain the static deformation (3.1).

Boulanger and Hayes (**7**) have shown that two axes, namely the 'acoustic' axes of the $\mathbb{B}^{-1}$-ellipsoid, exhibit remarkable properties with respect to the propagation of finite-amplitude homogeneous plane waves in such a deformed Mooney–Rivlin material. The acoustic axes are the only directions in which circularly polarized finite-amplitude homogeneous plane waves may propagate. Their directions $\mathbf{n}^\pm$ are defined in terms of the stretch ratios λ_1, λ_2, λ_3, and of the unit vectors $\mathbf{i}$ and $\mathbf{k}$, which are in the directions of the principal axes of $\mathbb{B}$ (or $\mathbb{B}^{-1}$) corresponding to λ_1 and λ_3, respectively. These directions are given by

$$\mathbf{n}^\pm = \cos\phi\mathbf{i} \pm \sin\phi\mathbf{k}, \text{ where } \cos\phi = \sqrt{\frac{\lambda_2^{-2} - \lambda_1^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}},\ \sin\phi = \sqrt{\frac{\lambda_3^{-2} - \lambda_2^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}}, \tag{3.3}$$

and are independent of the material parameters C and D. They lie along the normals to the planes of central circular sections of the $\mathbb{B}^{-1}$-ellipsoid (**12**, § 5.7). Their directions $\mathbf{n}^\pm$ also appear in the Hamilton cyclic decomposition of the $\mathbb{B}^{-1}$ tensor as

$$\mathbb{B}^{-1} = \lambda_2^{-2}\mathbf{1} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})(\mathbf{n}^+ \otimes \mathbf{n}^- + \mathbf{n}^- \otimes \mathbf{n}^+). \tag{3.4}$$

Now we consider a further deformation, possibly time-dependent, in which the particle at $\mathbf{x}$ moves to $\overline{\mathbf{x}}$ such that

$$\overline{\mathbf{x}} = \mathbf{x} + \epsilon\mathbf{u}(\mathbf{x}, t). \tag{3.5}$$

Here the displacement $\epsilon\mathbf{u}$ is a vector depending on $\mathbf{x}$ and on the time t, and ϵ is a small parameter. Throughout the paper, we neglect terms of second and higher orders in ϵ.

Now if the deformation gradient $\overline{\mathbb{F}}$, the strain tensor $\overline{\mathbb{B}}$, the invariants $\overline{I}$, $\overline{II}$, the pressure $\overline{p}$, and the stress tensor $\overline{\mathbb{T}}$ corresponding to the deformation (3.5) are expanded in powers of ϵ around their their value in the static state of homogeneous deformation, we get

$$\begin{aligned}
\overline{\mathbb{F}} &= \mathbb{F} + \epsilon\mathbb{F}^* + \cdots, & \overline{\mathbb{B}} &= \mathbb{B} + \epsilon\mathbb{B}^* + \cdots, \\
\overline{I} &= I + \epsilon I^* + \cdots, & \overline{II} &= II + \epsilon II^* + \cdots, \\
\overline{p} &= p + \epsilon p^* + \cdots, & \overline{\mathbb{T}} &= \mathbb{T} + \epsilon\mathbb{T}^* + \cdots.
\end{aligned} \tag{3.6}$$

Then the equations of motion (2.4) written for $\overline{\mathbb{T}}$ and $\overline{\mathbf{x}}$ take the form (**18**)

$$\operatorname{div} \mathbb{T}^* = \rho(\partial^2 \mathbf{u}/\partial t^2) \tag{3.7}$$

when we retain terms up to order ϵ. Explicitly, for the Mooney–Rivlin case, they are found to be (see also Belward (**13**), Hayes and Horgan (**19**) for the biaxial case)

$$\begin{aligned} -p^*_{,1} + C\lambda_1^2 u_{1,11} + (C\lambda_2^2 + D\lambda_1^{-2})u_{1,22} + (C\lambda_3^2 + D\lambda_1^{-2})u_{1,33} \\ -D\lambda_2^{-2}u_{2,12} - D\lambda_3^{-2}u_{3,13} = \rho\ddot{u}_1, \\ -p^*_{,2} + (C\lambda_1^2 + D\lambda_2^{-2})u_{2,11} + C\lambda_2^2 u_{2,22} + (C\lambda_3^2 + D\lambda_2^{-2})u_{2,33} \\ -D\lambda_1^{-2}u_{1,12} - D\lambda_3^{-2}u_{3,23} = \rho\ddot{u}_2, \\ -p^*_{,3} + (C\lambda_1^2 + D\lambda_3^{-2})u_{3,11} + (C\lambda_2^2 + D\lambda_3^{-2})u_{3,22} + C\lambda_3^2 u_{3,33} \\ -D\lambda_1^{-2}u_{1,13} - D\lambda_2^{-2}u_{2,23} = \rho\ddot{u}_3, \end{aligned} \tag{3.8}$$

where commas and dots denote differentiation with respect to position $\mathbf{x}$ and time t, respectively. Thus, for example, $u_{1,2} = \partial u_1/\partial y$.

Finally, the incompressibility constraint (2.2) yields (**18**)

$$u_{1,1} + u_{2,2} + u_{3,3} = 0. \tag{3.9}$$

4. Vibrations of complex exponential type

In order to solve the equations governing the incremental motion, we assume that the displacement $\epsilon\mathbf{u}$ and the incremental pressure ϵp^* are of the form

$$\mathbf{u} = \tfrac{1}{2}\{\mathbf{A}e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \quad p^* = \tfrac{1}{2}\{i\omega Q e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{4.1}$$

where $\mathbf{A} = \mathbf{A}' + i\mathbf{A}''$ is a complex vector, called the 'amplitude bivector' (**12**), ω is the real frequency, Q is a scalar, $\mathbf{S} = \mathbf{S}' + i\mathbf{S}''$ is the 'slowness bivector', and c.c. stands for 'complex conjugate'.

An ellipse may be associated with a bivector, as the ellipse for which the vector corresponding to the real part and the vector corresponding to the imaginary part are conjugate vectors. That is, to a bivector $\mathbf{D} = \mathbf{D}' + i\mathbf{D}''$ say, we associate the ellipse described by the point M such that $\mathbf{OM} = \mathbf{D}'\cos\theta + \mathbf{D}''\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

When the real and imaginary parts of $\mathbf{A}$ are parallel, that is, when the ellipse of $\mathbf{A}$ degenerates into a segment, the wave is linearly polarized along their common direction. Otherwise, the wave is elliptically polarized, and the ellipse of polarization is the ellipse of $\mathbf{A}$. The special case of circular polarization corresponds to the 'isotropy' of $\mathbf{A}$ (**16**)

$$\mathbf{A} \cdot \mathbf{A} = 0. \tag{4.2}$$

The planes defined by $\mathbf{S}' \cdot \mathbf{x} =$ constant are called planes of constant phase, and the planes defined by $\mathbf{S}'' \cdot \mathbf{x} =$ constant are the planes of constant amplitude. When the real part $\mathbf{S}'$ and the imaginary part $\mathbf{S}''$ of $\mathbf{S}$ are not parallel, the wave is said to be inhomogeneous. In that case, we introduce the 'directional ellipse' of the slowness bivector $\mathbf{S}$, defined as follows. Let $\widehat{\mathbf{m}}$ and $\widehat{\mathbf{n}}$ be unit vectors along the respective major and minor semi-axes of the ellipse of $\mathbf{S}$, and m be the aspect ratio of the ellipse of $\mathbf{S}$. Then the directional ellipse of the slowness bivector $\mathbf{S}$ is the ellipse of the 'propagation bivector' $\mathbf{C}$, defined by

$$\mathbf{C} = m\widehat{\mathbf{m}} + i\widehat{\mathbf{n}}, \quad \text{with} \quad \widehat{\mathbf{m}} \cdot \widehat{\mathbf{m}} = \widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} = 1, \quad \widehat{\mathbf{m}} \cdot \widehat{\mathbf{n}} = 0. \tag{4.3}$$

Then $\mathbf{S}$ may be written as (**16**) $\mathbf{S} = N\mathbf{C} = N(m\widehat{\mathbf{m}} + i\widehat{\mathbf{n}})$, where $N = N' + iN''$ is a complex number, called the 'complex scalar slowness'. Then the directions of propagation and attenuation are the directions of $\mathbf{S}'$ and $\mathbf{S}'$, respectively, given by

$$\mathbf{S}' = mN'\widehat{\mathbf{m}} - N''\widehat{\mathbf{n}}, \quad \mathbf{S}'' = mN''\widehat{\mathbf{m}} + N'\widehat{\mathbf{n}}. \tag{4.4}$$

Note that

$$\mathbf{S}' \cdot \mathbf{S}'' = (m^2 - 1)N'N'' \tag{4.5}$$

and, therefore, the planes of constant phase are orthogonal to the planes of constant amplitude either when $m = 1$ or when N is purely real or purely imaginary.

So, we may write solutions of the form (4.1) in terms of $\mathbf{C}$ as

$$\mathbf{u} = \tfrac{1}{2}\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \quad p^* = \tfrac{1}{2}\{i\omega NPe^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{4.6}$$

where $P = N^{-1}Q$. Now we substitute these expressions into (3.8) and (3.9).

First, equation (3.9) imposes the condition

$$\mathbf{A} \cdot \mathbf{C} = 0. \tag{4.7}$$

A geometrical interpretation (**12**, § 2.4) of this equation is that the orthogonal projection of the ellipse of the bivector $\mathbf{C}$ onto the plane of the bivector $\mathbf{A}$ is similar and similarly situated to the ellipse of $\mathbf{A}$, rotated through a quadrant. The amplitude and propagation bivectors are said to be 'orthogonal'. For homogeneous plane waves, the equation reduces to $\mathbf{A} \cdot \mathbf{n} = 0$ (where $\mathbf{n}$ is a real vector in the direction of propagation), which simply means that the polarization of the wave is transverse.

Next, the equations of motion (3.8) yield

$$-P\mathbf{C} + C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{A} + D[(\mathbf{C} \cdot \mathbf{C})\mathbb{B}^{-1}\mathbf{A} - (\mathbf{A} \cdot \mathbb{B}^{-1}\mathbf{C})\mathbf{C}] = \rho N^{-2}\mathbf{A}. \tag{4.8}$$

Taking the dot product of this last equation with $\mathbf{C}$, and using (4.7), yields

$$-P(\mathbf{C} \cdot \mathbf{C}) = 0. \tag{4.9}$$

In conclusion, the propagation of small-amplitude waves of complex exponential type in a homogeneously deformed Mooney–Rivlin material is governed by the following equations:

$$\begin{aligned} &-P\mathbf{C} + C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{A} + D[(\mathbf{C} \cdot \mathbf{C})\mathbf{1} - \mathbf{C} \otimes \mathbf{C}]\mathbb{B}^{-1}\mathbf{A} = \rho N^{-2}\mathbf{A}, \\ &\mathbf{A} \cdot \mathbf{C} = 0, \quad P(\mathbf{C} \cdot \mathbf{C}) = 0. \end{aligned} \tag{4.10}$$

Now, we treat in turn the case where $\mathbf{C}$ is isotropic ($\mathbf{C} \cdot \mathbf{C} = 0$) and the case where $\mathbf{C}$ is not isotropic ($\mathbf{C} \cdot \mathbf{C} \neq 0$).

5. Propagating evanescent solutions, $\mathbf{C}\cdot\mathbf{C}=0$

Here it is seen that corresponding to *any* isotropic bivector $\mathbf{C}$, there exists an infinity of linearly, circularly, and elliptically polarized inhomogeneous plane wave solutions.

First, we note that when $\mathbf{C}\cdot\mathbf{C}=0$, we have $m=1$ in (4.3) and we deduce from (4.5) that the planes of constant phase are orthogonal to the planes of constant amplitude.

Next, the equations (4.8) and (4.9) reduce to

$$\begin{aligned} &-P\mathbf{C}+C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{A}-D(\mathbf{A}\cdot\mathbb{B}^{-1}\mathbf{C})\mathbf{C}=\rho N^{-2}\mathbf{A},\\ &\mathbf{A}\cdot\mathbf{C}=0,\quad \mathbf{C}\cdot\mathbf{C}=0. \end{aligned}\tag{5.1}$$

Equations $(5.1)_{2,3}$ allow us to decompose $\mathbf{A}$ as (**12**, § 2.9)

$$\mathbf{A}=\alpha_1\mathbf{C}+\alpha_2\mathbf{C}\wedge\widetilde{\mathbf{C}},\tag{5.2}$$

where α_1, α_2 are real constants and $\widetilde{\mathbf{C}}$ is the complex conjugate of $\mathbf{C}$. Note that because $\mathbf{C}$ is isotropic, it is written as $\mathbf{C}=\widehat{\mathbf{m}}+i\widehat{\mathbf{n}}$ (equation (4.3) with $m=1$) and so $\mathbf{C}\wedge\widetilde{\mathbf{C}}=2i\widehat{\mathbf{n}}\wedge\widehat{\mathbf{m}}$ is parallel to a real vector. Substituting (5.2) in $(5.1)_1$ leads to two different types of solutions.

(i) *First type of solution: linearly and elliptically polarized waves*. Here the amplitude bivector $\mathbf{A}$ is given by $\mathbf{A}=\alpha_1\mathbf{C}+\alpha_2\mathbf{C}\wedge\widetilde{\mathbf{C}}$, $\alpha_2\neq 0$, where α_1, α_2 are real constants. The corresponding complex scalar slowness N is given by

$$\rho N^{-2}=C(\mathbf{C}\cdot\mathbb{B}\mathbf{C}),\tag{5.3}$$

and the incremental pressure is given by (4.6), where

$$P=-\alpha_1 D(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})-\alpha_2 D(\mathbf{C}\wedge\widetilde{\mathbf{C}})\cdot\mathbb{B}^{-1}\mathbf{C}.\tag{5.4}$$

Note that in this case, $\mathbf{A}\cdot\mathbf{A}=-\alpha_2^2(\mathbf{C}\cdot\widetilde{\mathbf{C}})^2\neq 0$, and thus the wave is not circularly polarized. When $\alpha_1=0$, the wave is linearly polarized in the direction of $\mathbf{C}\wedge\widetilde{\mathbf{C}}$. When $\alpha_1\neq 0$, the wave is elliptically polarized. Also, note that (5.3) can be written in terms of the slowness bivector $\mathbf{S}$ as $\rho=C\mathbf{S}\cdot\mathbf{B}\mathbf{S}$. The imaginary part of this equation yields $\mathbf{S}'\cdot\mathbb{B}\mathbf{S}''=0$, which means that the direction of the normal to the planes of equal phase and the direction of the normal to the planes of equal amplitude are conjugate directions with respect to the $\mathbb{B}$-ellipsoid. This condition has been established previously for linearly polarized finite-amplitude inhomogeneous plane waves propagating in a deformed Mooney–Rivlin material with an isotropic slowness bivector (**11**).

(ii) *Second type of solution: circularly polarized waves*. Here the amplitude bivector $\mathbf{A}$ is given by $\mathbf{A}=\alpha_1\mathbf{C}$, $\alpha_1\neq 0$, where α_1 is a real constant, and the incremental pressure is given by (4.6), where

$$P=\alpha_1[C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})-D(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})-\rho N^{-2}].\tag{5.5}$$

For this wave, the complex scalar slowness N is *arbitrary* (a similar situation was encountered by Boulanger and Hayes (**14**)); thus the displacement, which is the real part of $\epsilon\{\alpha_1\mathbf{C}\exp i\omega(N\mathbf{C}\cdot\mathbf{x}-t)\}$ is independent of the material constants C and D. Also, $\mathbf{A}\cdot\mathbf{A}=\alpha_1^2(\mathbf{C}\cdot\mathbf{C})=0$ and the wave is circularly polarized.

Because there is an infinity of choices for an isotropic bivector **C**, there is a triple infinity of propagation, attenuation, and polarization directions for linearly (type (i), $\alpha_1 = 0$), elliptically (type (i), $\alpha_1 \neq 0$), and circularly (type (ii)) polarized inhomogeneous plane waves of complex exponential type, provided the slowness bivector is isotropic. This is in sharp contrast to elliptically and circularly polarized *homogeneous* plane waves, which can only propagate along an acoustic axis (**7**).

Now we present explicit examples of the two types of solution corresponding to an isotropic propagation bivector **C**.

Example 1: *waves with an isotropic slowness bivector.* We choose an isotropic bivector **C** and write the corresponding two types of solutions. One set of solutions consists of elliptically polarized waves, the other set consists of circularly polarized waves travelling with an arbitrary speed. Let $\mathbf{C} = (3\mathbf{i} + 5i\mathbf{j} + 4\mathbf{k})/5$.

For waves of type (i), we have, using (5.2), $\mathbf{A} = (3\alpha_1 + 4i\alpha_2)\mathbf{i} + 5i\alpha_1\mathbf{j} + (4\alpha_1 - 3i\alpha_2)\mathbf{k}$, $\rho N^{-2} = C(9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2)/25$ and $P = -D[(9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2)\alpha_1 + 12i(\lambda_1^{-2} - \lambda_3^{-2})\alpha_2]/5$.

If the primary finite deformation is such that $9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2 > 0$, then N is real and we can write the following solution to the incremental equations of motion in a deformed Mooney–Rivlin material (3.8):

$$\begin{aligned}
u_1 &= e^{-\omega|N|y}\{3\alpha_1 \cos\omega[|N|(3x + 4z)/5 - t] - 4\alpha_2 \sin\omega[|N|(3x + 4z)/5 - t]\},\\
u_2 &= -5\alpha_1 e^{-\omega|N|y} \sin\omega[|N|(3x + 4z)/5 - t],\\
u_3 &= e^{-\omega|N|y}\{4\alpha_1 \cos\omega[|N|(3x + 4z)/5 - t] + 3\alpha_2 \sin\omega[|N|(3x + 4z)/5 - t]\},\\
p^* &= \tfrac{1}{5}D\omega|N|e^{-\omega|N|y}\{\alpha_1(9\lambda_1^{-2} - 25\lambda_2^{-2} + 16\lambda_3^{-2}) \sin\omega[|N|(3x + 4z)/5 - t]\\
&\quad + 12\alpha_2(\lambda_1^{-2} - \lambda_3^{-2}) \cos\omega[|N|(3x + 4z)/5 - t]\},
\end{aligned} \tag{5.6}$$

where $\omega, \alpha_1, \alpha_2$ are arbitrary ($\alpha_2 \neq 0$), and

$$|N| = 5\sqrt{\rho/[C(9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2)]}. \tag{5.7}$$

This wave travels in the direction of $3\mathbf{i} + 4\mathbf{k}$ and is attenuated in the y-direction. If $9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2 < 0$, then a solution is given by

$$\begin{aligned}
u_1 &= e^{-\omega|N|(3x+4z)/5}\{3\alpha_1 \cos\omega[-|N|y - t] - 4\alpha_2 \sin\omega[-|N|y - t]\},\\
u_2 &= -5\alpha_1 e^{-\omega|N|(3x+4z)/5} \sin\omega[-|N|y - t],\\
u_3 &= e^{-\omega|N|(3x+4z)/5}\{4\alpha_1 \cos\omega[-|N|y - t] + 3\alpha_2 \sin\omega[-|N|y - t]\},\\
p^* &= \tfrac{1}{5}D\omega|N|e^{-\omega|N|(3x+4z)/5}\{\alpha_1(9\lambda_1^{-2} - 25\lambda_2^{-2} + 16\lambda_3^{-2}) \sin\omega[-|N|y - t]\\
&\quad + 12\alpha_2(\lambda_1^{-2} - \lambda_3^{-2}) \cos\omega[-|N|y - t]\},
\end{aligned} \tag{5.8}$$

where $\omega, \alpha_1, \alpha_2$ are arbitrary ($\alpha_2 \neq 0$), and $|N|$ is again given by (5.7). This wave travels in the direction of $-\mathbf{j}$ and is attenuated in the direction of $3\mathbf{i} + 4\mathbf{k}$. In both cases (5.6) and (5.8), the wave propagates with speed $|N|^{-1}$ where $|N|$ is given by (5.7), and is elliptically polarized (two conjugate radii of the ellipse are $\alpha_1(3\mathbf{i} + 4\mathbf{k})$ and $5\alpha_1\mathbf{j} + \alpha_2(4\mathbf{i} - 3\mathbf{k})$).

For waves of type (ii), we have: $\mathbf{A} = \alpha_1(3\mathbf{i} + 5i\mathbf{j} + 4\mathbf{k})$, ρN^{-2} is arbitrary and $P = \alpha_1[C(9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2) - D(9\lambda_1^{-2} - 25\lambda_2^{-2} + 16\lambda_2^{-2}) - \rho N^{-2}]/5$. For N real, the corresponding solution is given by

$$\begin{aligned} u_1 &= 3\alpha_1 e^{-\omega N y} \cos\omega[N(3x + 4z)/5 - t], \\ u_2 &= -5\alpha_1 e^{-\omega N y} \sin\omega[N(3x + 4z)/5 - t], \\ u_3 &= 4\alpha_1 e^{-\omega N y} \cos\omega[N(3x + 4z)/5 - t], \\ p^* &= \tfrac{1}{5}\alpha_1 \omega N e^{-\omega N y}\{\rho N^{-2} - C(9\lambda_1^2 - 25\lambda_2^2 + 16\lambda_3^2) \\ &\quad + D(9\lambda_1^{-2} - 25\lambda_2^{-2} + 16\lambda_3^{-2})\} \sin\omega[N(3x + 4z)/5 - t], \end{aligned} \tag{5.9}$$

where α_1, ω and N are arbitrary. This wave travels in the direction of $3\mathbf{i} + 4\mathbf{k}$ with an arbitrary speed N^{-1}, is attenuated in the y-direction, and is circularly polarized (two orthogonal radii of the circle are $\alpha_1(3\mathbf{i} + 4\mathbf{k})$ and $5\alpha_1\mathbf{j}$). The displacement field $\epsilon\mathbf{u}$ does not depend on the constants C and D. It is easily checked that the solutions (5.6), (5.8) and (5.9) satisfy the general equations of motion (3.8).

6. Propagating evanescent solutions, $\mathbf{C} \cdot \mathbf{C} \neq 0$

In this section, we consider the case where the bivector $\mathbf{C}$ is not isotropic. A great variety of solutions is uncovered, and a systematic method of construction and classification for linearly, circularly, and elliptically polarized waves is presented.

When $\mathbf{C} \cdot \mathbf{C} \neq 0$, $(4.10)_3$ implies that $P = 0$ and therefore $p^* = 0$. Equation $(4.10)_1$ then reduces to

$$\{C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{1} + D[(\mathbf{C} \cdot \mathbf{C})\mathbf{1} - \mathbf{C} \otimes \mathbf{C}]\mathbb{B}^{-1}\}\mathbf{A} = \rho N^{-2}\mathbf{A}. \tag{6.1}$$

Following (**7**), we symmetrize this equation, using (4.7), to give the equivalent form

$$\mathbf{\Pi}[C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{1} + D(\mathbf{C} \cdot \mathbf{C})\mathbb{B}^{-1}]\mathbf{\Pi}\mathbf{A} = \rho N^{-2}\mathbf{A} \quad \text{and} \quad \mathbf{A} \cdot \mathbf{C} = 0, \tag{6.2}$$

where we have introduced the 'complex' projection operator $\mathbf{\Pi}$, defined by

$$\mathbf{\Pi} = \mathbf{1} - \frac{\mathbf{C} \otimes \mathbf{C}}{\mathbf{C} \cdot \mathbf{C}}. \tag{6.3}$$

This operator generalizes the 'real' projection $\mathbf{1} - \mathbf{n} \otimes \mathbf{n}$ upon the plane $\mathbf{n} \cdot \mathbf{x} = 0$ (**7**), and has the following properties: $\mathbf{\Pi}^2 = \mathbf{\Pi}$, $\mathbf{\Pi}\mathbf{C} = 0$ and $\mathbf{\Pi}\mathbf{A} = \mathbf{A}$ when $\mathbf{A} \cdot \mathbf{C} = 0$.

By inspection of (6.2), we see that solving the equations of motion, once $\mathbf{C}$ is prescribed, is equivalent to finding the eigenbivectors $\mathbf{A}$ of the tensor $\mathbf{\Pi}[C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{1} + D(\mathbf{C} \cdot \mathbf{C})\mathbb{B}^{-1}]\mathbf{\Pi}$ such that $\mathbf{A} \cdot \mathbf{C} = 0$, and their corresponding eigenvalues ρN^{-2}. This procedure is analogous to that used by Boulanger and Hayes in (**7**) for finite-amplitude homogeneous plane waves, with the replacement of their real vectors of propagation $\mathbf{n}$ and polarization $\mathbf{a}$, and their speed c with the bivectors $\mathbf{C}$ and $\mathbf{A}$, and the quantity N^{-1}, respectively.

First of all, we compute the possible eigenvalues ρN^{-2}.

6.1 *Secular equation*

The equations (6.2) admit solutions, provided that

$$\det(\mathbf{\Pi}[C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{1} + D(\mathbf{C}\cdot\mathbf{C})\mathbb{B}^{-1}]\mathbf{\Pi} - \rho N^{-2}\mathbf{1}) = 0. \tag{6.4}$$

Equation (6.4) is the classical secular equation for inhomogeneous plane waves. Because $\det\mathbf{\Pi} = 0$, it follows that $\rho N^{-2} = 0$ is one root of (6.4). The two other eigenvalues $\rho N_{\pm}^{-2}$ (say) are the roots of the quadratic

$$\begin{aligned}[]
[\rho N^{-2} - C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})]^2 - D[(\mathbf{C}\cdot\mathbf{C})\mathrm{tr}\,\mathbb{B}^{-1} - (\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})][\rho N^{-2} - C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})]& \\
+ D^2(\mathbf{C}\cdot\mathbf{C})(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) = 0.&
\end{aligned} \tag{6.5}$$

Explicitly, $\rho N_{\pm}^{-2}$ are given in terms of $\mathbf{C}$ alone as

$$\begin{aligned}
2\rho N_{\pm}^{-2} = {}& 2C(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) + D[(\mathbf{C}\cdot\mathbf{C})\mathrm{tr}\,\mathbb{B}^{-1} - (\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})] \\
& \pm D\sqrt{[(\mathbf{C}\cdot\mathbf{C})\mathrm{tr}\,\mathbb{B}^{-1} - (\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})]^2 - 4(\mathbf{C}\cdot\mathbf{C})(\mathbf{C}\cdot\mathbb{B}\mathbf{C})}.
\end{aligned} \tag{6.6}$$

These last quantities are the two eigenvalues corresponding to the amplitude bivectors which are orthogonal to $\mathbf{C}$. Equations (6.6) generalize the corresponding equations written for homogeneous waves by Boulanger and Hayes (**8**), by replacing their vector of propagation $\mathbf{n}$ and speed c by $\mathbf{C}$ and N^{-1}, respectively. Now we establish in turn the conditions for linear, circular, and elliptical polarization.

6.2 *Linearly-polarized waves,* $\mathbf{C}\cdot\mathbf{C} \neq 0$

Here we prove that linearly polarized inhomogeneous waves with a non-isotropic slowness bivector can propagate in a deformed Mooney–Rivlin material only when they are polarized in a principal direction.

When the amplitude bivector $\mathbf{A}$ is parallel to a real unit vector $\mathbf{a}$ (linear polarization), the incompressibility constraint (4.7) written as $\mathbf{a}\cdot\mathbf{S} = 0$ implies that the directions of propagation ($\mathbf{S}'$) and attenuation ($\mathbf{S}''$) are both orthogonal to $\mathbf{a}$. Hence, we write $\mathbf{a} = \alpha\mathbf{S}'\wedge\mathbf{S}''$, where α is such that $\mathbf{a}\cdot\mathbf{a} = 1$. The equations of motion (4.10) then reduce to

$$C(\mathbf{S}\cdot\mathbb{B}\mathbf{S})\mathbf{a} + D[(\mathbf{S}\cdot\mathbf{S})\mathbf{1} - \mathbf{S}\otimes\mathbf{S}]\mathbb{B}^{-1}\mathbf{a} = \rho\mathbf{a}, \quad \mathbf{a} = \alpha\mathbf{S}'\wedge\mathbf{S}'', \quad P = 0. \tag{6.7}$$

The dot product of $(6.7)_1$ by $\widetilde{\mathbf{S}} = \mathbf{S}' - i\mathbf{S}''$ yields

$$(\mathbf{S}\cdot\mathbf{S})(\mathbf{a}\cdot\mathbb{B}^{-1}\widetilde{\mathbf{S}}) - (\mathbf{S}\cdot\widetilde{\mathbf{S}})(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S}) = 0 \tag{6.8}$$

or, separating real and imaginary parts,

$$\begin{aligned}
(\mathbf{S}''\cdot\mathbf{S}'')(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S}') - (\mathbf{S}'\cdot\mathbf{S}'')(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S}'') = 0, \\
(\mathbf{S}'\cdot\mathbf{S}'')(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S}') - (\mathbf{S}'\cdot\mathbf{S}')(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S}'') = 0.
\end{aligned} \tag{6.9}$$

Using $(\mathbf{S}'\cdot\mathbf{S}')(\mathbf{S}''\cdot\mathbf{S}'') - (\mathbf{S}'\cdot\mathbf{S}'')^2 = \alpha^{-2}\mathbf{a}\cdot\mathbf{a} \neq 0$, we see that this homogeneous linear system of two equations admits only trivial solutions, that is, $\mathbf{a} \cdot \mathbb{B}^{-1}\mathbf{S}' = \mathbf{a} \cdot \mathbb{B}^{-1}\mathbf{S}'' = 0$. Thus $\mathbf{a}$ is parallel to $\mathbb{B}^{-1}\mathbf{S}' \wedge \mathbb{B}^{-1}\mathbf{S}''$, so that for some β,

$$\mathbf{a} = \beta(\mathbb{B}^{-1}\mathbf{S}' \wedge \mathbb{B}^{-1}\mathbf{S}'') = \beta\mathbb{B}(\mathbf{S}' \wedge \mathbf{S}'') = \alpha^{-1}\beta\mathbb{B}\mathbf{a}. \tag{6.10}$$

This last equality means that $\mathbf{a}$ is an eigenvector of $\mathbb{B}$. Therefore, provided $\mathbf{S}$ is not isotropic, linearly polarized inhomogeneous plane waves of complex exponential type can propagate in a deformed Mooney–Rivlin material only when polarized in a principal direction of the primary homogeneous deformation. Examples of such waves are given in section 6.4. When $\mathbf{S}$ (or equivalently $\mathbf{C}$) is isotropic, the inhomogeneous plane wave can be linearly polarized in non-principal directions (such as in example (5.6) when $\alpha_1 = 0$).

6.3 *Circularly polarized waves,* $\mathbf{C} \cdot \mathbf{C} \neq 0$

Here, we establish the condition for the propagation of circularly polarized inhomogeneous plane waves with a non-isotropic slowness bivector. From (4.2) and (6.1), the equations governing the propagation of such waves are

$$\begin{aligned} &C(\mathbf{C} \cdot \mathbb{B}\mathbf{C})\mathbf{A} + D(\mathbf{C} \cdot \mathbf{C})\mathbb{B}^{-1}\mathbf{A} - D(\mathbf{A} \cdot \mathbb{B}^{-1}\mathbf{C})\mathbf{C} = \rho N^{-2}\mathbf{A}, \\ &\mathbf{A} \cdot \mathbf{C} = 0, \quad \mathbf{A} \cdot \mathbf{A} = 0. \end{aligned} \tag{6.11}$$

Taking the dot product of $(6.11)_1$ by $\mathbf{A}$ yields $\mathbf{A} \cdot \mathbb{B}^{-1}\mathbf{A} = 0$ or, using (3.4),

$$\mathbf{A} \cdot \mathbf{n}^{\pm} = 0. \tag{6.12}$$

Therefore, $\mathbf{A}$ is orthogonal to $\mathbf{n}^{\pm}$. In other words, the planes of circular polarization for inhomogeneous plane waves with a non-isotropic slowness bivector are the planes of central circular section of the $\mathbb{B}^{-1}$-ellipsoid. This is the same situation as for homogeneous plane waves (**7**).

From $(6.11)_{2,3}$ and (6.12) it follows that the projection of the ellipse of $\mathbf{C}$ onto a plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid is a circle. Note that there is an infinity of such $\mathbf{C}$.

We consider the case where $\mathbf{A} \cdot \mathbf{n}^+ = 0$ and denote the corresponding isotropic amplitude bivector by $\mathbf{A}^{\theta}_{\odot}$. Noting that $\mathbf{n}^+$ is defined by (3.3), it follows that $\mathbf{A}^{\theta}_{\odot}$ is given by (**12**, § 2.2)

$$\mathbf{A}^{\theta}_{\odot} = \alpha e^{i\theta}(\mathbf{j} \wedge \mathbf{n}^+ + i\mathbf{j}), \quad \alpha, \theta \text{ real}. \tag{6.13}$$

Now, because $\mathbf{A}^{\theta}_{\odot}$, $\widetilde{\mathbf{A}}^{\theta}_{\odot}$, and $\mathbf{n}^+$ are linearly independent bivectors, the corresponding propagation bivector $\mathbf{C}^{\theta}_{\odot}$ (say) can be written as $\mathbf{C}^{\theta}_{\odot} = \lambda\mathbf{A}^{\theta}_{\odot} + \mu\widetilde{\mathbf{A}}^{\theta}_{\odot} + \nu\mathbf{n}^+$. From $(6.11)_2$ we have $\mathbf{C}^{\theta}_{\odot} \cdot \mathbf{A}^{\theta}_{\odot} = \mu\widetilde{\mathbf{A}}^{\theta}_{\odot} \cdot \mathbf{A}^{\theta}_{\odot} = 0$, and therefore $\mu = 0$. Then from (4.3), we have $\mathbf{C}^{\theta}_{\odot} \cdot \mathbf{C}^{\theta}_{\odot} = m^2 - 1 = \nu^2$, and therefore, ν is real and given by $\nu = \sqrt{m^2 - 1}$. Also, $\mathbf{C}^{\theta}_{\odot} \cdot \widetilde{\mathbf{C}}^{\theta}_{\odot} = m^2 + 1 = 2\lambda\widetilde{\lambda}\alpha^2 + \nu^2$, and so $|\lambda\alpha| = 1$. We conclude that $\mathbf{C}$ can be written as

$$\mathbf{C}^{\theta}_{\odot} = e^{i\theta}(\mathbf{j} \wedge \mathbf{n}^+ + i\mathbf{j}) + \sqrt{m^2 - 1}\mathbf{n}^+. \tag{6.14}$$

Using (6.13) and (6.14), we compute the eigenvalue $\rho(N^{\theta}_{\odot})^{-2}$ (say) from $(6.11)_1$ and find that it is given by

$$\rho(N^{\theta}_{\odot})^{-2} = \Lambda^{\theta}_{\odot}\big[C\Lambda^{\theta}_{\odot} + D\sqrt{m^2 - 1}\big], \tag{6.15}$$

where

$$\Lambda^{\theta}_{\odot} = \lambda_2^{-2}\sqrt{m^2 - 1} + e^{i\theta}\sqrt{(\lambda_2^{-2} - \lambda_1^{-2})(\lambda_3^{-2} - \lambda_2^{-2})}. \tag{6.16}$$

Example 2: *circularly polarized waves.* As an example of circularly polarized solutions, we consider the case where $\theta = 0$. In that case, we deduce from (6.13) to (6.16) the following solution to the equations of motion (3.8):

$$\begin{aligned} u_1 &= \alpha \sin\phi e^{-\omega N^0_\odot y} \cos\omega(N^0_\odot \mathbf{m}\cdot\mathbf{x} - t), \\ u_2 &= -\alpha e^{-\omega N^0_\odot y} \sin\omega(N^0_\odot \mathbf{m}\cdot\mathbf{x} - t), \\ u_3 &= -\alpha \cos\phi e^{-\omega N^0_\odot y} \cos\omega(N^0_\odot \mathbf{m}\cdot\mathbf{x} - t) \quad \text{and} \quad p^* = 0. \end{aligned} \tag{6.17}$$

Here α, ω and m are arbitrary, $\cos\phi$ and $\sin\phi$ are defined by (3.3), $N^0_\odot$ is given by (6.15), (6.16) with $\theta = 0$, and $\mathbf{m}$ is the vector defined by

$$\mathbf{m} = \mathbf{j}\wedge\mathbf{n}^+ + \sqrt{m^2-1}\mathbf{n}^+ = \big(\sin\phi + \sqrt{m^2-1}\cos\phi\big)\mathbf{i} - \big(\cos\phi - \sqrt{m^2-1}\sin\phi\big)\mathbf{k}.$$

Note that $N^0_\odot$ is real, and therefore the planes of constant phase are orthogonal to the planes of constant amplitude (see (4.5)).

The wave described by (6.17) is circularly polarized in the plane orthogonal to the acoustic axis $\mathbf{n}^+$, as would be the case for a homogeneous plane wave. The wave is attenuated in the y-direction and travels with speed $(mN^0_\odot)^{-1}$ in the direction of $\mathbf{m}$, which is not along an acoustic axis, in contrast to homogeneous waves.

6.4 *Elliptically polarized waves,* $\mathbf{C}\cdot\mathbf{C} \neq 0$

Now we consider elliptically polarized waves. In order to find the planes of polarization, we follow a procedure introduced by Boulanger and Hayes (**7**) for the propagation of finite-amplitude homogeneous plane waves in a deformed Mooney–Rivlin material. However, their method, which dealt with real vectors, is generalized here to the case of bivectors.

We know from section 6.3 that inhomogeneous plane waves (with a non-isotropic $\mathbf{S}$) are *not* circularly polarized when the projection of the ellipse of $\mathbf{C}$ upon the plane orthogonal to $\mathbf{n}^\pm$ is not a circle. Choosing $\mathbf{a}$ and $\mathbf{b}$, two unit vectors such that ($\mathbf{n}^\pm$, $\mathbf{a}$, $\mathbf{b}$) form an orthogonal triad, we decompose $\mathbf{C}$ as $\mathbf{C} = (\mathbf{C}\cdot\mathbf{a})\mathbf{a} + (\mathbf{C}\cdot\mathbf{b})\mathbf{b} + (\mathbf{C}\cdot\mathbf{n}^+)\mathbf{n}^+$, and see that we must have

$$(\mathbf{C}\cdot\mathbf{a})^2 + (\mathbf{C}\cdot\mathbf{b})^2 \neq 0 \quad \text{or} \quad \mathbf{C}\cdot\mathbf{C} - (\mathbf{n}^+\cdot\mathbf{C})^2 \neq 0. \tag{6.18}$$

This last inequality is equivalent to

$$\mathbf{n}^\pm \cdot \boldsymbol{\Pi}\mathbf{n}^\pm \neq 0. \tag{6.19}$$

Within this context, we seek the non-isotropic eigenbivectors $\mathbf{A}_\pm$ (say) of the tensor $\boldsymbol{\Pi}[C(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{1} + D(\mathbf{C}\cdot\mathbf{C})\mathbb{B}^{-1}]\boldsymbol{\Pi}$ such that $\mathbf{A}_\pm\cdot\mathbf{C} = 0$. We note that with the Hamilton cyclic decomposition (3.4) of the tensor $\mathbb{B}^{-1}$, we may write the tensor $\boldsymbol{\Pi}\mathbb{B}^{-1}\boldsymbol{\Pi}$ as

$$\boldsymbol{\Pi}\mathbb{B}^{-1}\boldsymbol{\Pi} = \lambda_2^{-2}\boldsymbol{\Pi} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})(\boldsymbol{\Pi}\mathbf{n}^+ \otimes \boldsymbol{\Pi}\mathbf{n}^- + \boldsymbol{\Pi}\mathbf{n}^- \otimes \boldsymbol{\Pi}\mathbf{n}^+). \tag{6.20}$$

Using (6.19), we construct the bivectors $\mathbf{H}^\pm$, given by

$$\mathbf{H}^\pm = \frac{\boldsymbol{\Pi}\mathbf{n}^\pm}{\sqrt{\mathbf{n}^\pm\cdot\boldsymbol{\Pi}\mathbf{n}^\pm}}, \quad \mathbf{H}^\pm\cdot\mathbf{H}^\pm = 1. \tag{6.21}$$

Note that $\mathbf{\Pi H}^{\pm} = \mathbf{H}^{\pm}$, so that we may write (6.20) as

$$\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi} = \lambda_2^{-2}\mathbf{\Pi} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})\sqrt{(\mathbf{n}^+ \cdot \mathbf{\Pi n}^+)(\mathbf{n}^- \cdot \mathbf{\Pi n}^-)}[\mathbf{H}^+ \otimes \mathbf{H}^- + \mathbf{H}^- \otimes \mathbf{H}^+].$$

From the definition (3.3) of $\mathbf{n}^{\pm}$ and (3.4), we note that $\mathbf{n}^+ \wedge \mathbf{n}^-$ (parallel to $\mathbf{j}$), $\mathbf{n}^+ + \mathbf{n}^-$ (parallel to $\mathbf{i}$), and $\mathbf{n}^+ - \mathbf{n}^-$ (parallel to $\mathbf{k}$) are orthogonal eigenvectors of $\mathbb{B}^{-1}$. Similarly, it can be checked that the eigenbivectors of the symmetric operator $\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi}$ are $\mathbf{C}$ (parallel to $\mathbf{H}^+ \wedge \mathbf{H}^-$) and the orthogonal bivectors $\mathbf{A}_{\pm}$ defined by

$$\mathbf{A}_{\pm} = \mathbf{H}^+ \pm \mathbf{H}^- = \frac{\mathbf{\Pi n}^+}{\sqrt{\mathbf{n}^+ \cdot \mathbf{\Pi n}^+}} \pm \frac{\mathbf{\Pi n}^-}{\sqrt{\mathbf{n}^- \cdot \mathbf{\Pi n}^-}}. \tag{6.22}$$

Now, because $\mathbf{\Pi A}_{\pm} = \mathbf{A}_{\pm}$, we can find the amplitude bivectors $\mathbf{A}$, solutions to (6.2), and orthogonal to the propagation bivector $\mathbf{C}$. They are the two bivectors $\mathbf{A}_{\pm}$ given by (6.22) and their corresponding eigenvalues $\rho N_{\pm}^{-2}$ are given by (6.6), or in this case, by

$$\begin{aligned}\rho N_{\pm}^{-2} = C(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + \frac{D}{2}(\mathbf{C} \cdot \mathbf{C})\Big\{&(\lambda_3^{-2} + \lambda_1^{-2}) \\ &+ (\lambda_3^{-2} - \lambda_1^{-2})\left[\frac{(\mathbf{n}^+ \cdot \mathbf{C})(\mathbf{n}^- \cdot \mathbf{C})}{\mathbf{C} \cdot \mathbf{C}} \mp \sqrt{(\mathbf{n}^+ \cdot \mathbf{\Pi n}^+)(\mathbf{n}^- \cdot \mathbf{\Pi n}^-)}\right]\Big\}.\end{aligned}$$

We note that

$$\begin{aligned}\rho(N_+^{-2} - N_-^{-2}) &= -D(\mathbf{C} \cdot \mathbf{C})(\lambda_3^{-2} - \lambda_1^{-2})\sqrt{(\mathbf{n}^+ \cdot \mathbf{\Pi n}^+)(\mathbf{n}^- \cdot \mathbf{\Pi n}^-)}, \\ &= -D(\mathbf{C} \cdot \mathbf{C})(\lambda_3^{-2} - \lambda_1^{-2})\sqrt{\left[1 - \frac{(\mathbf{n}^+ \cdot \mathbf{C})^2}{\mathbf{C} \cdot \mathbf{C}}\right]\left[1 - \frac{(\mathbf{n}^- \cdot \mathbf{C})^2}{\mathbf{C} \cdot \mathbf{C}}\right]}.\end{aligned} \tag{6.23}$$

When the bivector $\mathbf{C}$ and the complex quantities $N_{\pm}^{-1}$ are replaced by $\mathbf{n}$ (direction of propagation) and c_1, c_2 (speeds) for homogeneous waves, this equation reduces to $\rho(c_2^2 - c_1^2) = D(\lambda_3^{-2} - \lambda_1^{-2}) \sin\phi^+ \sin\phi^-$ (**7**), where $\phi^{\pm}$ are the angles between $\mathbf{n}$ and $\mathbf{n}^{\pm}$. This in turn is reminiscent of the 'law of the product of the two sines' (*la loi du produit des deux sinus*) for the propagation of light through a biaxial crystal in classical linear optics, established empirically by Biot (**20**) in 1818 and theoretically by Fresnel (**21**) in 1821.

From (6.23) and (6.19) we see that here, the eigenvalues are distinct and therefore neither $\mathbf{A}_+$ nor $\mathbf{A}_-$ is isotropic (**12**, § 3.2). This means that, as expected, the waves corresponding to these amplitudes are elliptically polarized (**16**) (and not circularly polarized). In this connection, it may be recalled that (**15**) *finite*-amplitude inhomogeneous plane waves of complex exponential type can only be *linearly* polarized (and not elliptically polarized), when they propagate in an incompressible elastic material. Here this restriction is removed because we are dealing with *small*-amplitude inhomogeneous plane waves.

Finally, using (6.1) and the fact that $\mathbf{A}_{\pm} \cdot \mathbf{A}_{\pm} \neq 0$, we can write $\rho N_{\pm}^{-2}$ in terms of $\mathbf{C}$ and $\mathbf{A}_{\pm}$ as

$$\rho N_{\pm}^{-2} = C(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + D(\mathbf{C} \cdot \mathbf{C})\frac{\mathbf{A}_{\pm} \cdot \mathbb{B}^{-1}\mathbf{A}_{\pm}}{\mathbf{A}_{\pm} \cdot \mathbf{A}_{\pm}}. \tag{6.24}$$

For homogeneous plane waves, these expressions reduce to results established by Boulanger and

Hayes (**7**): on replacing $N_{\pm}^{-2}$ with their squared wave speeds c_1, c_2, and the bivectors $\mathbf{C}$ and $\mathbf{A}_{\pm}$ by their real unit vectors $\mathbf{n}$, $\mathbf{a}$ and $\mathbf{b}$, respectively, the expressions are transformed into $\rho c_1^2 = C(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + D(\mathbf{a} \cdot \mathbb{B}^{-1}\mathbf{a})$ and $\rho c_2^2 = C(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + D(\mathbf{b} \cdot \mathbb{B}^{-1}\mathbf{b})$.

We now write down explicit examples of elliptically polarized waves.

Example 3: *elliptically polarized waves.* In this example, we present waves propagating in a principal direction with attenuation in another principal direction. It is seen that for one solution, the wave is elliptically polarized in the plane of the slowness bivector, while for the other solution, the wave is linearly polarized in the direction normal to the plane of the slowness bivector.

We take $\mathbf{C} = m\mathbf{i} + i\mathbf{j}$ with $m > 1$.

For waves corresponding to the eigenvalue ρN_{+}^{-2}, we have for the amplitude $\mathbf{A}_{+} = \alpha(\mathbf{i} + im\mathbf{j})$, with eigenvalue: $\rho N_{+}^{-2} = C(m^2\lambda_1^2 - \lambda_2^2) + D(m^2\lambda_2^{-2} - \lambda_1^{-2})$, so that the corresponding solution is given, for N_{+} real, by

$$
\begin{aligned}
u_1 &= \alpha e^{-\omega N_{+} y} \cos \omega(mN_{+}x - t),\\
u_2 &= -\alpha m e^{-\omega N_{+} y} \sin \omega(mN_{+}x - t),\\
u_3 &= 0 \quad \text{and} \quad p^* = 0,
\end{aligned}
\tag{6.25}
$$

where ω, α and m are arbitrary ($m > 1$), and

$$
N_{+} = \sqrt{\rho/[C(m^2\lambda_1^2 - \lambda_2^2) + D(m^2\lambda_2^{-2} - \lambda_1^{-2})]}. \tag{6.26}
$$

For waves corresponding to the eigenvalue ρN_{-}^{-2}, we find the amplitude to be: $\mathbf{A}_{-} = \alpha\mathbf{k}$, with eigenvalue: $\rho N_{-}^{-2} = C(m^2\lambda_1^2 - \lambda_2^2) + D(m^2 - 1)\lambda_3^{-2}$, and the corresponding solution is given, for N_{-} real, by

$$
u_1 = 0, \quad u_2 = 0, \quad u_3 = \alpha e^{-\omega N_{-} y} \cos \omega(mN_{-}x - t), \quad p^* = 0, \tag{6.27}
$$

where ω, α and m are arbitrary ($m > 1$), and

$$
N_{-} = \sqrt{\rho/[C(m^2\lambda_1^2 - \lambda_2^2) + D(m^2 - 1)\lambda_3^{-2}]}. \tag{6.28}
$$

The waves described by (6.25) and (6.27) propagate in the x-direction with speed $(mN_{\pm})^{-1}$, where m is prescribed and $N_{\pm}$ is given by (6.26) and (6.28), respectively. They are attenuated in the y-direction. The wave given by (6.25) is elliptically polarized in the (x, y)-plane (two conjugate radii of the ellipse are $\alpha\mathbf{i}$ and $m\alpha\mathbf{j}$); the wave given by (6.27) is linearly polarized in the z-direction.

Example 4: *elliptically polarized waves.* Our next example is a wave solution propagating along an acoustic axis with attenuation in the direction orthogonal to the acoustic axes.

Let $\mathbf{C} = \sqrt{2}\mathbf{n}^{+} + i\mathbf{j} = \sqrt{2}\cos\phi\mathbf{i} + i\mathbf{j} + \sqrt{2}\sin\phi\mathbf{k}$, where 2ϕ is the angle between the acoustic axes, defined by (3.3). After some calculation, we write another explicit solution to the incremental equations of motion (3.8) as

$$\begin{aligned} u_1 &= \left[-1 \pm \frac{1-2\cos 2\phi}{\sqrt{\cos 4\phi}}\right]\cos\phi\, e^{-\omega N_\pm y}\sin\omega(\sqrt{2}N_\pm \mathbf{n}^+\cdot\mathbf{x}-t),\\ u_2 &= -\sqrt{2}\left[1 \pm \frac{\cos 2\phi}{\sqrt{\cos 4\phi}}\right]e^{-\omega N_\pm y}\cos\omega(\sqrt{2}N_\pm \mathbf{n}^+\cdot\mathbf{x}-t),\\ u_3 &= \left[-1 \mp \frac{1+2\cos 2\phi}{\sqrt{\cos 4\phi}}\right]\sin\phi\, e^{-\omega N_\pm y}\sin\omega(\sqrt{2}N_\pm \mathbf{n}^+\cdot\mathbf{x}-t) \quad \text{and} \quad p^*=0, \end{aligned} \tag{6.29}$$

where ω is arbitrary, ϕ and $\mathbf{n}^+$ are defined by (3.3), and

$$\rho N_\pm^{-2} = C(2\lambda_2^{-4}-\lambda_2^2) + \tfrac{1}{2}D\big\{\lambda_3^{-2}+\lambda_1^{-2}+(\lambda_3^{-2}-\lambda_1^{-2})\big[2\cos 2\phi \mp \sqrt{\cos 4\phi}\big]\big\}. \tag{6.30}$$

The waves described by (6.29) propagate in the direction of the acoustic axis $\mathbf{n}^+$ with speed $(\sqrt{2}N_\pm)^{-1}$, where $N_\pm$ is given by (6.30). They are attenuated in the y-direction, and are elliptically polarized.

Example 5: *elliptically polarized waves.* Our last examples are inhomogeneous plane waves for which the planes of constant phase are not orthogonal to the planes of constant amplitude.

Let $\mathbf{C} = m\mathbf{p} + i\mathbf{q}$ ($m > 1$), where

$$\mathbf{p} = \cos\theta\mathbf{i} + \sin\theta\mathbf{k}, \quad \mathbf{q} = \sin\theta\mathbf{i} - \cos\theta\mathbf{k}, \quad 0 < \theta < \pi/2. \tag{6.31}$$

The corresponding slowness bivector $\mathbf{S} = N\mathbf{C}$ has real and imaginary parts given by

$$\mathbf{S}' = mN'\mathbf{p} - N''\mathbf{q}, \quad \mathbf{S}'' = mN''\mathbf{p} + N'\mathbf{q}. \tag{6.32}$$

It is found from (6.22) that the amplitude bivectors $\mathbf{A}_\pm$ are given by

$$\mathbf{A}_+ = \alpha(m\mathbf{q} - i\mathbf{p}), \quad \mathbf{A}_- = \alpha\mathbf{j}. \tag{6.33}$$

Corresponding to $\mathbf{A}_+$, we have the inhomogeneous wave

$$\mathbf{u} = \alpha e^{-\omega\mathbf{S}''\cdot\mathbf{x}}[m\mathbf{q}\cos\omega(\mathbf{S}'\cdot\mathbf{x}-t) + \mathbf{p}\sin\omega(\mathbf{S}'\cdot\mathbf{x}-t)]. \tag{6.34}$$

The directions of propagation and attenuation are those of $\mathbf{S}'$ and $\mathbf{S}''$, given by (6.32), where N' and N'' are the real and imaginary parts of N, which is given from (6.24) by

$$\begin{aligned} \rho N^{-2} &= C[m^2(\mathbf{p}\cdot\mathbb{B}\mathbf{p}) - (\mathbf{q}\cdot\mathbb{B}\mathbf{q})] + D\frac{m^2-1}{m^2+1}[m^2(\mathbf{q}\cdot\mathbb{B}^{-1}\mathbf{q}) - (\mathbf{p}\cdot\mathbb{B}^{-1}\mathbf{p})]\\ &\quad + 2im\left[C(\mathbf{p}\cdot\mathbb{B}\mathbf{q}) - D\frac{m^2-1}{m^2+1}(\mathbf{p}\cdot\mathbb{B}^{-1}\mathbf{q})\right]. \end{aligned} \tag{6.35}$$

This wave propagates in the direction of $\mathbf{S}'$ with speed $|\mathbf{S}'|^{-1}$, is attenuated in the direction of $\mathbf{S}''$, and is elliptically polarized (two conjugate radii of the ellipse are $m\mathbf{p}$ and $\mathbf{q}$).

Corresponding to $\mathbf{A}_-$, we have the wave

$$\mathbf{u} = \alpha\mathbf{j}e^{-\omega\mathbf{S}''\cdot\mathbf{x}}\cos\omega(\mathbf{S}'\cdot\mathbf{x}-t). \tag{6.36}$$

Here, the respective directions of propagation and attenuation are those of $\mathbf{S}'$ and $\mathbf{S}''$, given by (4.4), where N' and N'' are the real and imaginary parts of N, which is given from (6.24) by

$$\rho N^{-2} = C[m^2(\mathbf{p}\cdot\mathbb{B}\mathbf{p}) - (\mathbf{q}\cdot\mathbb{B}\mathbf{q})] + D(m^2-1)\lambda_2^{-2} + 2im(\mathbf{p}\cdot\mathbb{B}\mathbf{q}). \tag{6.37}$$

This wave propagates in the direction of $\mathbf{S}'$ with speed $|\mathbf{S}'|^{-1}$, is attenuated in the direction of $\mathbf{S}''$, and is linearly polarized in the y-direction. In this connection, it may be noted that *finite*-amplitude linearly polarized inhomogeneous plane waves of complex exponential type can propagate in a homogeneously deformed Mooney–Rivlin material only when the planes of constant phase (orthogonal to $\mathbf{S}'$) are orthogonal with the planes of constant amplitude (orthogonal to $\mathbf{S}''$) (**11**). Here, from (6.32), we have $\mathbf{S}'\cdot\mathbf{S}'' = (m^2-1)N'N'' \neq 0$, and these planes are not orthogonal to each other. This is possible because we are dealing with *small*-amplitude inhomogeneous waves only.

7. Summary and concluding remarks

We have found all possible inhomogeneous small-amplitude motions of complex exponential type that can be superimposed upon the large homogeneous static deformation of a Mooney–Rivlin material. The results are summarized in Tables 1 and 2, where the solutions of exponential sinusoidal form are listed for inhomogeneous plane waves with an isotropic and a non-isotropic bivector $\mathbf{C}$, respectively. In the last column of the tables, it is recalled whether or not each motion is possible with a finite amplitude, and the corresponding reference is given. In the second column, the equation numbers of corresponding examples in this paper are given. Thus for instance, line 1.3 reads as follows: a small-amplitude circularly polarized inhomogeneous plane wave of complex exponential type with isotropic propagation bivector $\mathbf{C}$ can propagate in a homogeneously deformed Mooney–Rivlin material; for this wave, the amplitude bivector $\mathbf{A}$ is isotropic and orthogonal to $\mathbf{C}$; an example of such solution is given by equation (5.9); also note that $\mathbf{A}$ is parallel to $\mathbf{C}$ and that the complex scalar slowness N is arbitrary; finally, such a finite-amplitude wave solution is not possible in a homogeneously deformed Mooney–Rivlin material (see (**15**)).

Through the use of bivectors, *all* inhomogeneous solutions of complex exponential type for the problem of small deformations superposed on a large static triaxial strain of a Mooney–Rivlin incompressible hyperelastic material were obtained systematically. A great diversity and richness of solutions was uncovered, using the Directional Ellipse method (**16**).

For inhomogeneous waves with an isotropic slowness bivector (section 5), any direction of polarization or plane of polarization is permitted for linear or elliptical polarization, respectively; also, for circular polarization the complex scalar slowness can be arbitrarily prescribed.

For inhomogeneous waves with a non-isotropic slowness bivector (section 6), the secular equation was established and solved (section 6.1), generalizing the secular equation for homogeneous waves (**8**). Then it was seen that such linearly polarized waves (section 6.2) can only propagate along one of the principal axes of the finite static stretch; that circularly polarized waves (section 6.3) can only be polarized in one of the two planes of central circular section of the $\mathbb{B}^{-1}$-ellipsoid, but can propagate in directions other than the directions of the acoustic axes of this ellipsoid; and that all elliptically polarized wave solutions (section 6.4) can be obtained by generalizing a method introduced in (**7**) to the consideration of bivectors.

Also, in contrast to the case of *finite*-amplitude inhomogeneous plane waves (**11**, **15**), it was seen (sections 5 and 6.4) that elliptical polarization is possible, and that for waves linearly polarized in a

Table 1 Inhomogeneous plane waves, $\mathbf{C}$ isotropic ($\mathbf{C}\cdot\mathbf{C}=0$)

Polarization type	Displacement	Notes	Finite case?
1.1 Linear	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\wedge\widetilde{\mathbf{A}}=0$. Example: (5.6) with $\alpha_1=0$	$\mathbf{A}=\alpha_2\mathbf{C}\wedge\widetilde{\mathbf{C}}$	yes (**11**)
1.2 Elliptical	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\cdot\mathbf{A}\neq 0$. Example: (5.6) with $\alpha_1\neq 0$	$\mathbf{A}=\alpha_1\mathbf{C}+\alpha_2\mathbf{C}\wedge\widetilde{\mathbf{C}}$	no (**15**)
1.3 Circular	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\cdot\mathbf{A}=0$. Example: (5.9)	$\mathbf{A}=\alpha_1\mathbf{C}$, N is arbitrary	

Table 2 Inhomogeneous plane waves, $\mathbf{C}$ non-isotropic ($\mathbf{C}\cdot\mathbf{C}\neq 0$)

Polarization type	Displacement	Notes	Finite case?
2.1 Linear	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\wedge\widetilde{\mathbf{A}}=0$. Examples: (6.27), (6.36)	$\mathbf{A}$ along principal axis of basic strain	only when $\mathbf{S}'\cdot\mathbf{S}''=0$ ($\mathbf{S}=N\mathbf{C}$) (**11**)
2.2 Elliptical	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\cdot\mathbf{A}\neq 0$. Examples: (6.25), (6.29), (6.34)	$(\mathbf{n}^{\pm}\cdot\mathbf{C})^2\neq\mathbf{C}\cdot\mathbf{C}$	no (**15**)
2.3 Circular	$\{\mathbf{A}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}+\text{c.c.}\}$, $\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{A}\cdot\mathbf{A}=0$. Example: (6.17)	$\mathbf{A}\cdot\mathbf{n}^{\pm}=0$	

principal direction (section 6.2 and Example 5), the planes of constant phase need not be orthogonal to the planes of constant amplitude.

Finally, it is of interest to note that the results established here can easily be specialized to the case of a *biaxial* pure homogeneous static prestrain, simply by taking $\lambda_1=\lambda_2=\lambda$ (say), and $\lambda_3=\mu=\lambda^{-2}$ (say). In that case, the tensor $\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi}$ may be written as (**7**)

$$\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi}=\lambda^{-2}\mathbf{\Pi}-(\lambda^{-2}-\mu^{-2})\mathbf{\Pi}\mathbf{k}\otimes\mathbf{\Pi}\mathbf{k}. \tag{7.1}$$

It follows that the computation of the amplitude bivectors and corresponding complex scalar slownesses is greatly simplified, and that the results of Boulanger and Hayes (**14**) are directly recovered from the analysis presented here.

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