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The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity

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In an attempt to describe cork-pulling, we model a cork as an incompressible rubber-like material and consider that it is subject to a helical shear deformation superimposed onto a shrink fit and a simple torsion. It turns out that this deformation field provides an insight into the possible appearance of secondary deformation fields for special classes of materials. We also find that these latent deformation fields are woken up by normal stress differences. We present some explicit examples based on the neo-Hookean, the generalized neo-Hookean and the Mooney–Rivlin forms of the strain-energy density. Using the simple exact solution found in the neo-Hookean case, we conjecture that it is advantageous to accompany the usual vertical axial force by a twisting moment, in order to extrude a cork from the neck of a bottle efficiently. Then we analyse departures from the neo-Hookean behaviour by exact and asymptotic analyses. In that process, we are able to give an elegant and analytic example of secondary (or latent) deformations in the framework of nonlinear elasticity.

Keywords: nonlinear elasticity; semi-inverse method; boundary-value problems; secondary fields; cork-pulling

1. Introduction

Rubbers and elastomers are highly deformable solids, which have the remarkable property of preserving their volume through any deformation. This permanent isochoricity can be incorporated into the equations of continuum mechanics through the concept of an *internal constraint*, here the constraint of *incompressibility*. Mathematically, the formulation of the constraint of incompressibility has led to the discovery of several exact solutions in isotropic finite elasticity, most notably to the controllable or universal solutions of Rivlin and co-workers (e.g. Rivlin 1948). Subsequently, Ericksen (1954) examined the problem of finding all such solutions. He found that there are no controllable finite deformations in

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isotropic *compressible* elasticity, except for homogeneous deformations (Ericksen 1955). The impact of that result on the theory of nonlinear elasticity was quite important and long-lasting, and, for many years, a palpable pessimism reigned about the possibility of finding exact solutions at all for compressible elastic materials. Then Currie & Hayes (1981) showed that one could obtain interesting classes of exact solutions, beyond the homogeneous universal deformations, if attention was restricted to certain special classes of compressible materials. A string of results about the search for exact solutions in nonlinear elasticity followed. Now a long list exists for classes of exact solutions, which are universal only relative to some special strain-energy functions (for a recent presentation of such classes, see Fu & Ogden (2001)). These solutions can help us to understand the structure of the theory of nonlinear elasticity and to complement the celebrated solutions of Rivlin.

In the same vein, some recent efforts focused on determining the maximal strain energy for which a certain deformation field, fixed *a priori*, is admissible. This is a sort of *inverse problem*: find the elastic materials (i.e. the functional form of the strain-energy function) for which a given deformation field is controllable (i.e. for which the deformation is a solution to the equilibrium equations in the absence of body forces). A classic example illustrating such an approach is obtained by considering deformations of *anti-plane shear* type. Knowles (1976) shows that a non-trivial (non-homogeneous) equilibrium state of anti-plane shear is not always (universally) admissible, not only for compressible solids (as expected from Ericksen's result) but also for incompressible solids (Horgan (1995) gives a survey of anti-plane shear deformations in nonlinear elasticity). Only for a special class of incompressible materials (inclusive of the so-called 'generalized neo-Hookean materials') is an anti-plane shear deformation controllable.

Let us consider, for example, the case of an elastic material filling the annular region between two coaxial cylinders, with the following boundary-value problem: hold fixed the outer cylinder and pull the inner cylinder by applying a tension in the axial direction. It is well established that a solution to this problem, valid for every incompressible isotropic elastic solid, is obtained by assuming *a priori* that the deformation field is a pure axial shear. Now consider the corresponding problem for *non-coaxial* cylinders, thereby losing the axial symmetry. Then it is clear that we cannot expect the material to deform as prescribed by a pure axial shear deformation. Knowles's result tells us that now the boundary-value problem can be solved with a general anti-plane deformation (not axially symmetric) *only for a subclass* of incompressible isotropic elastic materials. Of course, this restriction does not mean that, for a generic material, it is not possible to deform the annular material as prescribed by our boundary conditions, but rather that, in general, these lead to a deformation field that is more complex than an anti-plane shear. Hence, we also expect *secondary* in-plane deformations.

The theory of non-Newtonian fluid dynamics has generated a substantial literature about secondary flows (e.g. Fosdick & Serrin 1973). In solid mechanics, it seems that only Fosdick & Kao (1978) and Mollica & Rajagopal (1997) produced some significant and beautiful examples of secondary deformation fields for the non-coaxial cylinders problem, although this topic is clearly of fundamental importance not only from a theoretical point of view but also for technical applications.

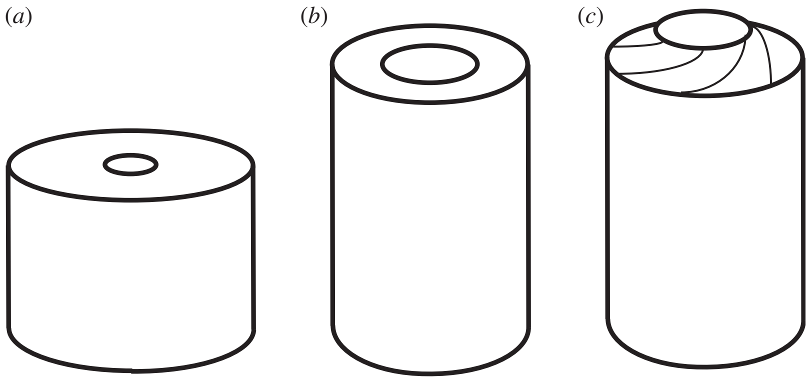

Figure 1. (a) Shrink fit of an elastic tube, followed by the combination of (b) simple torsion and (c) helical shear. (The figure is not drawn to scale among the various deformations.)

In this paper, we consider a complex deformation field in incompressible isotropic elasticity, to point out by an explicit example the situations just evoked and to elaborate on their possible impact on solid mechanics. Our deformation field takes advantage of the radial symmetry; therefore, we find it convenient to visualize it by considering an elastic cylinder.

Let us imagine that a corkscrew has been driven through a cork (the cylinder) in a bottle. The inside of the bottleneck is the outer rigid cylinder and the idealization of the gallery carved out by the corkscrew constitutes the inner coaxial rigid cylinder. Our first deformation is purely radial, originated from the introduction of the cork into the bottleneck and then completed when the corkscrew penetrates the cork (a so-called *shrink-fit problem*, which here is a source of elastic residual stresses). We call A and B the respective inner and outer radii of the cork in the reference configuration, and $r_1 > A$ and $r_2 < B$ their respective current counterparts. Then we follow with a *simple torsion* combined with a *helical shear*, in order to model pulling the cork out of the bottleneck in the presence of a contact force. Figure 1 sketches this deformation.

Of course, we are aware of the shortcomings of our modelling with respect to the description of a 'real' cork-pulling problem, because no cork is an infinitely long cylinder, nor is a corkscrew perfectly straight. In addition, traditional corks made from bark are anisotropic (honeycomb-shaped mesoscopic structure) and possess the remarkable (and little-known) property of having an infinitesimal Poisson ratio equal to zero (see the review article by Gibson *et al.* (1981)). However, we note that *polymer corks* have appeared on the world wine market; they are made of elastomers, for which incompressible isotropic elasticity seems like a reasonable framework (indeed, the documentation of these synthetic wine stoppers indicates that they lengthen during the sealing process). We hope that this study provides a first step towards a nonlinear alternative to the linear elasticity testing protocols presented in the international standard ISO 9727. We also note that low-cost *shock absorbers* often consist of a moving metal cylinder, glued to the inner face of an elastomeric tube, the outer face of which is glued to a fixed metal cylinder (Hill 1975).

The plan of the paper is as follows. Section 2 is devoted to the derivation of the governing equations and to a detailed description of the boundary-value problem. In §3 we specialize the analysis to the *neo-Hookean strain-energy density* and find the corresponding exact solution. We use it to show that it is advantageous to add a twisting moment to an axial force when extruding a cork from a bottle. The neo-Hookean strain-energy density is linear with respect to the first principal invariant of the Cauchy–Green strain tensor. It is much used in finite elasticity theory, although it poorly captures the basic features of rubber behaviour (Saccomandi 2004). We thus investigate the consequence of departing from that strain-energy density. First, in §4 we consider the *generalized neo-Hookean strain-energy density*—nonlinear with respect to the first principal invariant of the Cauchy–Green strain tensor—to show that, in this case, torsion is explicitly present in the solution for the axial shear displacement, but it is a second-order dependence. Next, in §5 we consider the *Mooney–Rivlin strain-energy density*—linear with respect to the first and second principal invariants of the Cauchy–Green strain tensor—and find that it is also possible to obtain an exact solution to our boundary-value problem. Its expression is too cumbersome to manipulate and we resort to a small-parameter asymptotic expansion from the neo-Hookean case. Section 6 concludes the paper with some remarks on the limitations of the semi-inverse method.

2. Basic equations

Consider a long hollow cylindrical tube composed of an incompressible isotropic nonlinearly elastic material. At rest, the tube is in the region

$$A \leq R \leq B, \qquad 0 \leq \Theta \leq 2\pi \quad \text{and} \quad -\infty \leq Z \leq \infty, \tag{2.1}$$

where (R, Θ, Z) are the cylindrical coordinates associated with the undeformed configuration, and A and B are the inner and outer radii of the tube, respectively.

(a) *Equilibrium equations*

Consider the general deformation obtained by the combination of radial dilatation, helical shear and torsion as

$$r = r(R), \qquad \theta = \Theta + g(R) + \tau Z \quad \text{and} \quad z = \lambda Z + w(R), \tag{2.2}$$

where (r, θ, z) are the cylindrical coordinates in the deformed configuration; τ is the amount of torsion; and λ is the stretch ratio in the Z-direction. Here, g and w are yet unknown functions of R only. (The classic case of pure torsion corresponds to $w = g = 0$; for instance, see Ogden (1997) or Atkin & Fox (2005).)

Hidden inside (2.2) is the *shrink-fit deformation*

$$r = r(R), \qquad \theta = \Theta \quad \text{and} \quad z = \lambda Z, \tag{2.3}$$

which is (2.2) without any torsion or helical shear $(\tau = g = w \equiv 0)$.

The physical components of the deformation gradient $\boldsymbol{F}$ and its inverse $\boldsymbol{F}^{-1}$ are then

$$\begin{bmatrix} r' & 0 & 0 \\ rg' & r/R & r\tau \\ w' & 0 & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r\lambda/R & 0 & 0 \\ rw'\tau - rg'\lambda & r'\lambda & -rr'\tau \\ -rw'/R & 0 & rr'/R \end{bmatrix}, \tag{2.4}$$

respectively. Note that we used the incompressibility constraint in order to compute $\boldsymbol{F}^{-1}$; it states that det $\boldsymbol{F}=1$, so that

$$r' = \frac{R}{\lambda r}. \tag{2.5}$$

In our first deformation, the cylindrical tube is pressed into a cylindrical cavity with inner radius $r_1 > A$ and outer radius $r_2 < B$. It follows by integration of equation (2.5) that

$$r(R) = \sqrt{\frac{R^2}{\lambda} + \alpha}, \tag{2.6}$$

where now

$$\alpha = \frac{B^2 r_1^2 - A^2 r_2^2}{B^2 - A^2} \quad \text{and} \quad \lambda = \frac{B^2 - A^2}{r_2^2 - r_1^2}. \tag{2.7}$$

We compute the physical components of the left Cauchy–Green strain tensor $\mathsf{B} \equiv \boldsymbol{F}\boldsymbol{F}^t$ from (2.4) and find its first three principal invariants $I_1 \equiv \operatorname{tr} \mathsf{B}$, $I_2 \equiv (\det \mathsf{B})\operatorname{tr}(\mathsf{B}^{-1})$ and $I_3 \equiv \det \mathsf{B}$ as

$$\begin{aligned} I_1 &= (r')^2 + (rg')^2 + (r/R)^2 + (r\tau)^2 + \lambda^2 + (w')^2, \\ I_2 &= (r\lambda/R)^2 + (rw'\tau - rg'\lambda)^2 + (rw'/R)^2 + (R/r)^2 + (1/\lambda)^2 + (R\tau/\lambda)^2 \end{aligned} \tag{2.8}$$

and, of course, $I_3 = 1$.

For a general incompressible hyperelastic solid, the Cauchy stress tensor T is related to the strain through

$$\mathsf{T} = -p\boldsymbol{I} + 2W_1\mathsf{B} - 2W_2\mathsf{B}^{-1}, \tag{2.9}$$

where p is the Lagrange multiplier introduced by the incompressibility constraint; $W = W(I_1, I_2)$ is the strain-energy density; and $W_i \equiv \partial W/\partial I_i$. Having computed $\mathsf{B}^{-1} \equiv (\boldsymbol{F}^t)^{-1}\boldsymbol{F}^{-1}$ from (2.4), we find that the components of T are

$$\begin{aligned} T_{rr} &= -p + 2W_1(r')^2 - 2W_2[(r\lambda/R)^2 + (rw'\tau - rg'\lambda)^2 + (rw'/R)^2], \\ T_{\theta\theta} &= -p + 2W_1[(rg')^2 + (r/R)^2 + (r\tau)^2] - 2W_2(R/r)^2, \\ T_{zz} &= -p + 2W_1[\lambda^2 + (w')^2] - 2W_2[(1/\lambda)^2 + (R\tau/\lambda)^2], \\ T_{r\theta} &= 2W_1(rr'g') - 2W_2(w'\tau - g'\lambda)R, \\ T_{rz} &= 2W_1(r'w') - 2W_2[rRg'\tau - rRw'\tau^2/\lambda - rw'/(\lambda R)] \quad \text{and} \\ T_{\theta z} &= 2W_1(rw'g' + r\lambda\tau) + 2W_2(r'R\tau). \end{aligned} \tag{2.10}$$

Finally, the equilibrium equations in the absence of body forces are as follows: div $\mathsf{T}=0$; for fields depending only on the radial coordinate as shown here, they reduce to

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}+\frac{T_{rr}-T_{\theta\theta}}{r}=0,\qquad \frac{\mathrm{d}T_{r\theta}}{\mathrm{d}r}+\frac{2}{r}T_{r\theta}=0 \quad\text{and}\quad \frac{\mathrm{d}T_{rz}}{\mathrm{d}r}+\frac{1}{r}T_{rz}=0. \tag{2.11}$$

(b) Boundary conditions

Now consider the inner face of the tube. We assume that it is subject to a vertical pull,

$$T_{rz}(A)=T_0^A \quad\text{and}\quad T_{r\theta}(A)=0, \tag{2.12}$$

say. Then by integrating the second and third equations of equilibrium $(2.11)_{2,3}$, we find that

$$T_{rz}(r)=\frac{r_1}{r}T_0^A \quad\text{and}\quad T_{r\theta}(r)=0. \tag{2.13}$$

The outer face of the tube (in contact with the glass in the cork/bottle problem) remains fixed, so that

$$w(B)=0,\qquad g(B)=0 \quad\text{and}\quad T_{rr}(B)=T_0, \tag{2.14}$$

say. In addition to the axial traction applied on its inner face, the tube is subject to a resultant axial force N (say) and a resultant moment M (say),

$$N=\int_0^{2\pi}\int_{r_1}^{r_2}T_{zz}r\,\mathrm{d}r\,\mathrm{d}\theta \quad\text{and}\quad M=\int_0^{2\pi}\int_{r_1}^{r_2}T_{\theta z}r^2\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.15}$$

Note that the traction T_0 of (2.14) is not arbitrary but is instead determined by the *shrink-fit pre-deformation* (2.3), by requiring that $N=0$ when $T_0^A=\tau=g=w\equiv 0$. (This process is detailed in §3.) Therefore, T_0 is connected with the stress field experienced by the cork when it is introduced in the bottleneck.

In the rest of the paper we aim at presenting results in dimensionless form. To this end, we normalize the strain-energy density W and the Cauchy stress tensor T with respect to μ, the infinitesimal shear modulus; hence, we introduce $\bar{W}$ and $\bar{\mathsf{T}}$ defined as

$$\bar{W}=\frac{W}{\mu} \quad\text{and}\quad \bar{\mathsf{T}}=\frac{\mathsf{T}}{\mu}. \tag{2.16}$$

Similarly, we introduce the following non-dimensional variables:

$$\eta=\frac{A}{B},\qquad \bar{R}=\frac{R}{B},\qquad \bar{r}_i=\frac{r_i}{B},\qquad \bar{w}=\frac{w}{B},\qquad \bar{\alpha}=\frac{\alpha}{B^2}\quad\text{and}\quad \bar{\tau}=B\tau, \tag{2.17}$$

so that $\eta\leq\bar{R}\leq 1$. Also, we find from (2.7) that

$$\bar{\alpha}=\frac{\bar{r}_1^2-\eta^2\bar{r}_2^2}{1-\eta^2} \quad\text{and}\quad \lambda=\frac{1-\eta^2}{\bar{r}_2^2-\bar{r}_1^2}. \tag{2.18}$$

Turning to our cork or shock absorber problems, we imagine that the inner metal cylinder is introduced into a pre-existing cylindrical cavity (this precaution ensures a one-to-one correspondence of the material points between the reference and the current configurations). In our upcoming numerical simulations, we take $A=B/10$, so that $\eta=0.1$; we consider that the outer radius is shrunk by 10%, $r_2=0.9B$, and that the inner radius is doubled, $r_1=2A$; finally, we apply a traction, the magnitude of which is half the infinitesimal shear modulus: $|T_0^A|=\mu/2$. This gives

$$\bar{\alpha}\simeq 3.22\times 10^{-2}, \qquad \lambda\simeq 1.286 \quad \text{and} \quad \bar{T}_0^A=-0.5. \tag{2.19}$$

At this point, it is possible to state clearly our main observation. A first glance at the boundary conditions, in particular at the requirements that g be zero on the outer face of the tube, gives the expectation that $g\equiv 0$ everywhere is a solution to our boundary-value problem, at least for some simple forms of the constitutive equations. In what follows, we find that, for the so-called 'neo-Hookean' solids, $g\equiv 0$ is indeed a solution, whether there is a torsion τ or not. However, if the solid is not neo-Hookean, then it is necessary that $g\neq 0$ when $\tau\neq 0$, and the picture becomes more complex. For this reason, we classify the following question as 'purely academic': *which is the most general strain-energy density for which it is possible to solve the above boundary-value problem with* $g\equiv 0$? Indeed, there is no 'real-world' material, the behaviour of which is ever going to be described *exactly* by that strain-energy density (supposing that it exists). Instead, a more pertinent issue to raise for 'real-world applications' is whether we are able to evaluate the importance of latent (secondary) stress fields, because they are bound to be woken up (triggered) by the deformation.

3. Neo-Hookean materials

First, we consider the special strain-energy density that generates the class of neo-Hookean materials, namely

$$W=(I_1-3)/2, \quad \text{so that} \quad 2W_1=1 \quad \text{and} \quad W_2=0. \tag{3.1}$$

Note that here and hereafter, we use the non-dimensional quantities introduced previously, from which we drop the overbar for convenience. Hence, the components of the (non-dimensional) stress field in a neo-Hookean material reduce to

$$\begin{aligned} T_{rr}&=-p+(r')^2, & T_{\theta\theta}&=-p+(rg')^2+(r/R)^2+(r\tau)^2,\\ T_{zz}&=-p+\lambda^2+(w')^2, & T_{r\theta}&=rr'g',\\ T_{rz}&=r'w' \quad \text{and} & T_{\theta z}&=rg'w'+r\lambda\tau. \end{aligned} \tag{3.2}$$

Substituting into (2.13), we find that

$$w'=\lambda r_1 T_0^A/R \quad \text{and} \quad g'=0 \tag{3.3}$$

and by integration, using (2.14), we have

$$w=\lambda r_1 T_0^A \ln R \quad \text{and} \quad g=0. \tag{3.4}$$

In figure 2*a*, we present a rectangle in the tube at rest, which is delimited by $0.1\leq R\leq 1.0$ and $0.0\leq Z\leq 1.0$. Then it is subject to the deformation

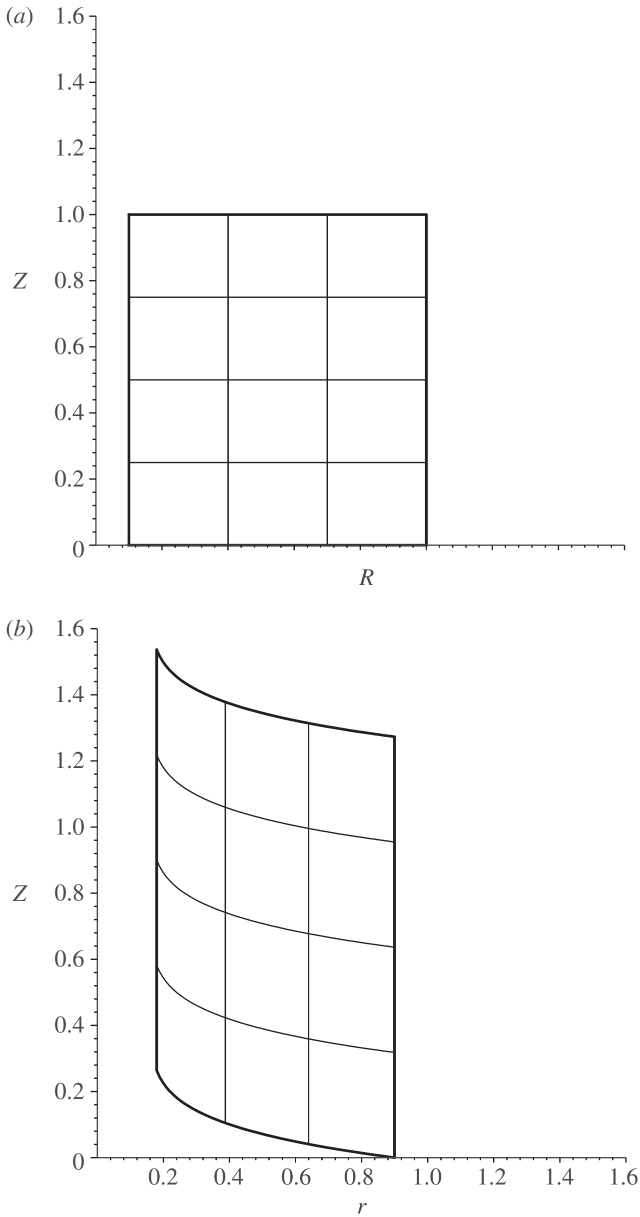

Figure 2. (a,b) Pulling on the inside face of a neo-Hookean tube. Here the vertical axis is the symmetry axis of the tube.

corresponding to the numerical values of (2.19). To generate figure 2b, we computed the resulting shape for a neo-Hookean tube, using (2.2), (2.6) and (3.4).

Now that we know the full deformation field (see (2.2) and (3.4)), we can compute $T_{rr} - T_{\theta\theta}$ from (3.2) and deduce T_{rr} by integration of $(2.11)_1$, with initial condition $(2.14)_3$. Then the other field quantities follow from the rest of (3.2).

Finally, we find in turn that

$$T_{rr} = \frac{1}{2\lambda}\left\{\ln\frac{\lambda r_2^2 R^2}{R^2+\alpha\lambda} + (R^2-1)\left[\frac{\alpha}{r_2^2(R^2+\alpha\lambda)} + \tau^2\right]\right\} + T_0,$$

$$T_{\theta\theta} = T_{rr} + \left(\frac{R^2}{\lambda}+\alpha\right)\left(\frac{1}{R^2}+\tau^2\right) - \frac{R^2}{\lambda(R^2+\alpha\lambda)} \quad \text{and} \tag{3.5}$$

$$T_{zz} = T_{rr} + \lambda^2\left(1+\frac{r_1^2(T_0^A)^2}{R^2}\right) - \frac{R^2}{\lambda(R^2+\alpha\lambda)}$$

(where we used the identity $1+\alpha\lambda = \lambda r_2^2$, see (2.6) with $R=1$) and that

$$T_{r\theta} = 0, \qquad T_{rz} = \frac{r_1}{\sqrt{\frac{R^2}{\lambda}+\alpha}}\, T_0^A \quad \text{and} \quad T_{\theta z} = \lambda\tau\sqrt{\frac{R^2}{\lambda}+\alpha}. \tag{3.6}$$

The constant T_0 is fixed by the shrink-fit pre-deformation (2.3), imposing that $N=0$ when $\tau = g = w = T_0^A \equiv 0$, or

$$(T_0+\lambda^2)(1-\eta^2) + \frac{1}{\lambda}\int_\eta^1 \left[\ln\frac{\lambda r_2^2 R^2}{R^2+\alpha\lambda} + \frac{\alpha(R^2-1)}{r_2^2(R^2+\alpha\lambda)} - \frac{2R^2}{R^2+\alpha\lambda}\right] R\,\mathrm{d}R = 0. \tag{3.7}$$

Using (3.7) and (2.15), (3.5) and (3.6), we find the following expressions for the resultant moment:

$$M = \pi\left(r_2^4 - r_1^4\right)\lambda\tau/2 \tag{3.8}$$

and for the axial force

$$N = 2\pi\lambda r_1^2 |\ln\eta| \left(T_0^A\right)^2 - \frac{\pi}{4}\left(r_2^2 - r_1^2\right)^2 \tau^2. \tag{3.9}$$

We now have a clear picture of the response of a neo-Hookean solid to the deformation (2.2), with the boundary conditions of §2*b*. First, we saw that here the contribution $g(R)$ is not required for the azimuthal displacement, whether there is a torsion τ or not. Also, if a moment $M \neq 0$ is applied, then the tube suffers an amount of torsion $\tau \neq 0$ proportional to M. On the other hand, if the tube is pulled by the application of an axial force only ($N \neq 0$) and no moment ($M=0$), then $\tau=0$ and no azimuthal shear occurs at all.

When we try to apply our results to the extrusion of a cork from the neck of a bottle, the following remarks seem to be relevant. From the elementary theory of Coulomb friction, it is known that the pulled cork starts to move when, in modulus, the friction force exerted on the neck surface is equal to the normal force times the coefficient of static friction. In our case, this means that

$$\sqrt{|T_{rz}(1)|^2 + |T_{r\theta}(1)|^2} = f_S |T_{rr}(1)| = f_S |T_0|, \tag{3.10}$$

where f_S is the coefficient of static friction. Using (2.12) and (2.13), we find that the elements of the l.h.s. of equality (3.10) are

$$T_{rz}(1) = (r_1/r_2)\, T_0^A \quad \text{and} \quad T_{r\theta}(1) = 0. \tag{3.11}$$

Now, our main concern is to understand whether it is better to apply a moment $M\neq 0$ when uncorking a bottle, than to pull only. To address this question, we note that the l.h.s. of equality (3.10) increases when $|T_0^A|$ increases; on the other hand, combining (3.8) and (3.9), we have

$$\left(T_0^A\right)^2 = \frac{\left[N + \frac{1}{\pi\lambda^2(r_1^2+r_2^2)^2} M^2\right]}{(2\pi\lambda r_1^2|\ln\eta|)}. \tag{3.12}$$

It is now clear that, for a fixed value of T_0^A, in the case $M\neq 0$, it is necessary to apply an axial force, the intensity of which is less than the one in the case $M=0$. Moreover, equation (3.12) shows that $\left(T_0^A\right)^2$ grows linearly with N but quadratically with M. With respect to efficient cork-pulling, the conclusion is that adding a twisting moment to a given pure axial force is more advantageous than solely increasing the vertical pull. Moreover, we observe that a moment is applied by using a lever and this is always more convenient from an energetic point of view.

Recall that we made several simplifying assumptions to reach these results: not only infinite axial length, incompressibility and isotropy, but also the choice of a truly special strain-energy density. In §§4 and 5 we depart from the neo-Hookean model.

4. Generalized neo-Hookean materials

As a first broadening of the neo-Hookean strain-energy density (3.1), we consider *generalized neo-Hookean materials*, for which the strain-energy density is a nonlinear function of the first invariant I_1 only,

$$W = \hat{W}(I_1), \tag{4.1}$$

say. To gain access to the Cauchy stress components in this context, it suffices to take $W_2=0$ and $W_1=\hat{W}'$ in equations (2.10). In particular, $T_{r\theta}=2rr'g'\hat{W}'$ and the integrated equation of equilibrium $(2.13)_2$ gives $g'=0$. By integrating, with (2.12) as an initial value, we find that

$$g \equiv 0. \tag{4.2}$$

Hence, just as in the neo-Hookean case, azimuthal shear can be avoided altogether, whether there is a torsion τ or not. We are left with an equation for the axial shear, namely $(2.13)_1$, which can be written as

$$2\hat{W}'(I_1)w'(R) = \frac{\lambda r_1}{R} T_0^A. \tag{4.3}$$

Obviously, the same steps as those taken for neo-Hookean solids may be followed here for any given strain-energy density (4.1), but now by resorting to a numerical treatment. Horgan & Saccomandi (2003*a*) show, through some specific examples of hardening generalized neo-Hookean solids, how rapidly involved the analysis becomes, even when there is only helical shear and no shrink fit.

Instead, we simply point out some striking differences between our present situation and the neo-Hookean case. We remark that I_1 is of the form $(2.8)_1$ at $g\equiv 0$, i.e.

$$I_1 = \lambda^2 + \frac{R^2}{\lambda(R^2+\alpha\lambda)} + \left(\frac{R^2}{\lambda} + \alpha\right)\left(\frac{1}{R^2} + \tau^2\right) + [w'(R)]^2. \tag{4.4}$$

It follows that (4.3) is a *nonlinear* differential equation for w', in contrast to the neo-Hookean case. Another contrast is that the axial shear w is now intimately coupled with the torsion parameter τ, and that this dependence is a *second-order effect* (τ appears above as τ^2).

A similar problem where the azimuthal shear has not been ignored, but the axial shear has been considered null, i.e. $w\equiv 0$, has recently been considered by Wineman (2005).

5. Mooney–Rivlin materials

In this section, we specialize the general equations of §2 to the Mooney–Rivlin form of the strain-energy density, which in its non-dimensional form reads

$$W = \frac{I_1 - 3 + m(I_2 - 3)}{2(1+m)}, \quad \text{so that} \quad 2W_1 = \frac{1}{1+m} \quad \text{and} \quad 2W_2 = \frac{m}{1+m}, \tag{5.1}$$

where $m>0$ is a material parameter, distinguishing the Mooney–Rivlin material from the neo-Hookean material (3.1), and also allowing a dependence on the second principal strain invariant I_2, in contrast to the generalized neo-Hookean solids of §4.

Then the integrated equations of equilibrium (2.13) can be written as

$$\begin{aligned} (R + m\tau^2 r^2 R + mr^2/R)w' - (m\tau\lambda r^2 R)g' &= (1+m)\lambda r_1 T_0^A \quad \text{and} \\ (m\tau\lambda)w' - (1+m\lambda^2)g' &= 0. \end{aligned} \tag{5.2}$$

First, we ask ourselves if it is possible to avoid torsion during the pulling of the inner face. Taking $\tau=0$ above gives

$$(R + mr^2/R)w' = (1+m)\lambda r_1 T_0^A \quad \text{and} \quad g' = 0. \tag{5.3}$$

It follows that here it is indeed possible to solve our boundary-value problem. We find

$$w = \lambda r_1 T_0^A \frac{\lambda(1+m)}{2(\lambda+m)} \ln\left[\frac{m\alpha\lambda + (\lambda+m)R^2}{m\alpha\lambda + (\lambda+m)}\right] \quad \text{and} \quad g = 0. \tag{5.4}$$

However, if $\tau\neq 0$, then it is necessary that $g\neq 0$, otherwise $(5.2)_2$ gives $w'=0$ while $(5.2)_1$ gives $w'\neq 0$, a contradiction. This constitutes the first departure from the neo-Hookean and generalized neo-Hookean behaviours: *torsion* ($\tau\neq 0$) *is necessarily accompanied by azimuthal shear* ($g\neq 0$).

In the case $\tau\neq 0$, we introduce the function $\Lambda=\Lambda(R)$ defined as

$$\Lambda(R) = (R + mr^2/R)(1+m\lambda^2) + m\tau^2 r^2 R \tag{5.5}$$

(recall that $r=r(R)$ is given explicitly in (2.6)). We then solve the system (5.2) for w' and g' as

$$w' = (1+m)(1+m\lambda^2)\lambda \frac{T_0^A}{\Lambda(R)} r_1 \quad \text{and} \quad g' = m(1+m)\lambda^2 \frac{T_0^A}{\Lambda(R)} \tau r_1, \tag{5.6}$$

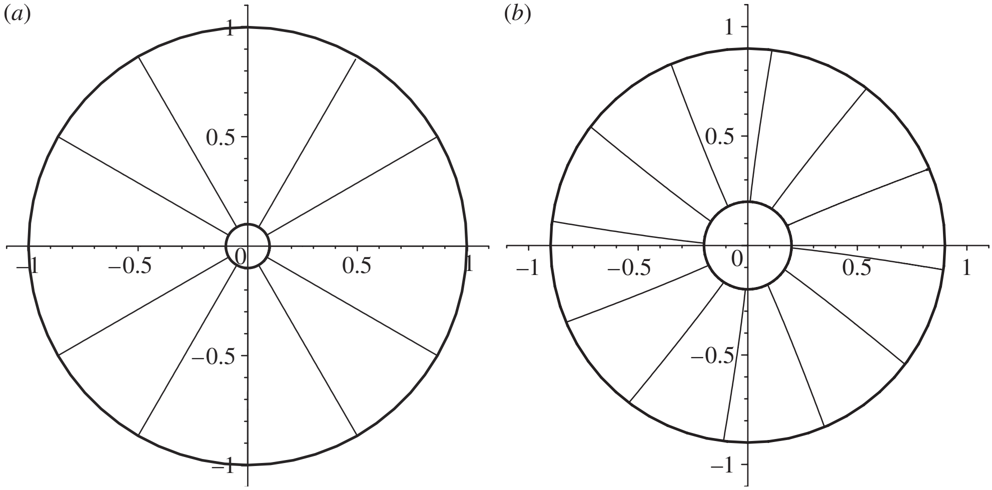

Figure 3. (a,b) Pulling on the inside face of a Mooney–Rivlin tube, with a clockwise torsion.

making clear the link between g and τ. Thus, for the Mooney–Rivlin material, the azimuthal shear g is a *latent* mode of deformation; it is *woken up* by any amount of torsion τ. Recall that, at first sight, the azimuthal shear component of the deformation (2.2) seemed inessential to satisfy the boundary conditions, especially in view of the boundary condition $g(1)=0$. However, a non-zero W_2 term in the constitutive equation clearly couples the effects of a torsion and an azimuthal shear, as displayed explicitly by the presence of τ in expression (5.6) for g'.

It is perfectly possible to integrate equations (5.6) in the general case, but to save space we do not reproduce the resulting long expressions. With them, we generated the deformation field picture of figure 3a,b. There we took the numerical values of (2.17) for α, λ, and T_0^A; we took a Mooney–Rivlin solid with $m=5.0$; we imposed a torsion of amount $\tau=0.5$; and we looked at the deformation field in the plane $Z=1$ (reference configuration) and $z=\lambda$ (current configuration).

Although the secondary fields appear to be slight in the picture, they are nonetheless truly present and cannot be neglected. To show this, we consider a perturbation method to obtain simpler solutions and to understand the effect of the coupling, by taking m small. Then integrating (5.6), we find at the first order that

$$\begin{aligned}\frac{w}{r_1 T_0^A} &\simeq (1+m)\lambda \ln R - \frac{1}{2} m[\tau^2 R^2 + 2(1+\tau^2\alpha\lambda)\ln R - \alpha\lambda/R^2 - \tau^2 + \alpha\lambda],\\ \frac{g}{r_1 T_0^A} &\simeq \lambda^2 \tau m \ln R.\end{aligned} \tag{5.7}$$

Hence, the secondary field g exists even for a nearly neo-Hookean solid (if m is small, then g is of order m). Interestingly, we also note that the azimuthal shear g in (5.7) varies in a homogeneous and linear manner with respect to the torsion parameter τ and in a quadratic manner with respect to the axial stretch λ, showing that the presence of this secondary deformation field cannot be neglected when the effects of both the prestress and the torsion are taken into account. To complete the picture, we use the first-order approximations

$$2W_1 \simeq 1-m \quad \text{and} \quad 2W_2 \simeq m, \tag{5.8}$$

to obtain the stress field as

$$\begin{aligned}
T_{rr} &\simeq -p + (1-m)(r')^2 - m\Big\{(r\lambda/R)^2 + [(r\tau)^2 + (r/R)^2]\big(\lambda r_1 T_0^A\big)^2/R^2\Big\},\\
T_{\theta\theta} &\simeq -p + (1-m)[(r/R)^2 + (r\tau)^2] - m(R/r)^2,\\
T_{zz} &\simeq -p + \big(\lambda T_0^A r_1\big)^2\left[(1+2m\lambda^2)\frac{1}{R^2} - \frac{2}{R}\left(\frac{\tau^2 r^2}{R} + \frac{r^2}{R^3} + \frac{\lambda^2}{R} - \frac{1}{2R}\right)m\right]\\
&\quad + (1-m)\lambda^2 - m[(1/\lambda)^2 + (R\tau/\lambda)^2],\\
T_{r\theta} &\simeq rr'g' - m\lambda r_1 T_0^A \tau,\\
T_{rz} &\simeq (1-m)(r'w') + m\lambda r_1 T_0^A[rR\tau^2/\lambda + r/(\lambda R)]/R,\\
T_{\theta z} &\simeq (1-m)r\lambda\tau + \lambda r r_1 T_0^A g'/R + m(r'R\tau).
\end{aligned} \tag{5.9}$$

Using this stress field, it is straightforward, but it is long and cumbersome to derive the analogue for a Mooney–Rivlin solid with a small m of relation (3.12) (which was established for neo-Hookean solids). However, nothing truly new is gained from these complex formulae with respect to the simple neo-Hookean case, and we do not pursue this aspect any further.

6. Conclusion

In non-Newtonian fluid mechanics and turbulence theory, the existence of shear-induced normal stresses on planes transverse to the direction of shear is at the root of some important phenomena occurring in the flow of fluid down pipes of non-circular cross section (Fosdick & Serrin 1973). In other words, pure parallel flows in tubes without axial symmetry are possible only when we consider the classical theory of Navier–Stokes equations, the linear theory of turbulence or tubes of circular cross section.

In nonlinear elasticity theory, similar phenomena are reported. Hence, Fosdick & Kao (1978) and Mollica & Rajagopal (1997) show that, for general incompressible isotropic materials, an anti-plane shear deformation of a cylinder with non-axial symmetric cross section causes a secondary in-plane deformation field, owing to normal stress differences. Horgan & Saccomandi (2003*b*) give a detailed discussion of how the anti-plane shear deformation field couples with the in-plane deformation field in a generalized neo-Hookean solid.

The appearance of what we called here *latent deformations* is quite general and common. For example, it is known in compressible nonlinear elasticity that pure torsion is possible only in a special class of materials, but we know that torsion plus a radial displacement is possible in all compressible isotropic elastic materials (Polignone & Horgan 1991). (Here we signal that 'possible in all materials' is not equivalent to 'universal', because the corresponding radial deformation differs from one material to another.)

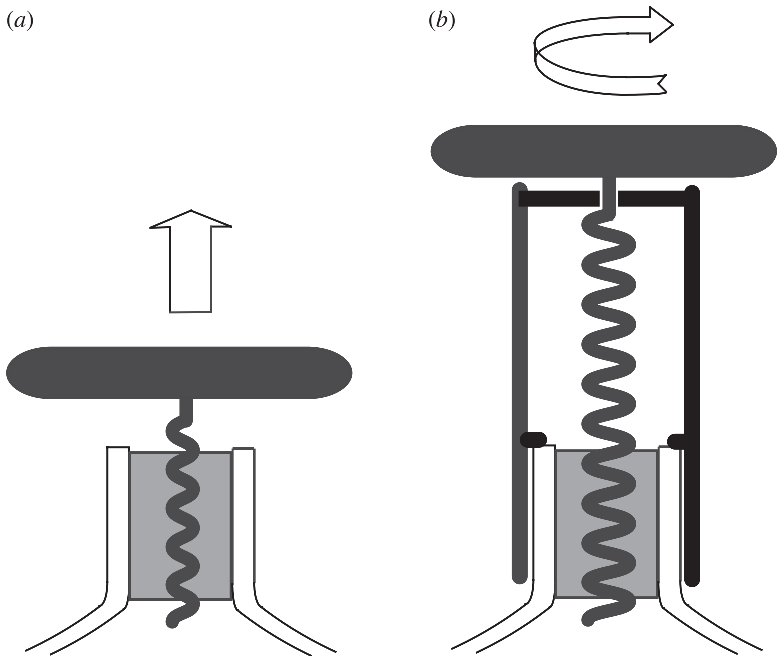

Figure 4. There are two main types of corkscrews: (*a*) one that relies on pulling only and (*b*) another that adds a twist to the cork-pulling action. The analysis developed in this paper indicates that the second type is more efficient.

In this paper, we give an example where axial symmetry holds, where the boundary conditions suggest that an axial shear deformation field is sufficient to solve the boundary-value problem, and where, nevertheless, the normal stress difference wakes up a latent azimuthal shear deformation. Moreover, since we are able to find some explicit exact solutions by some perturbation techniques, we are able to evaluate the *importance* of the latent deformation. Indeed, we show that, if a certain constitutive parameter m (distinguishing a neo-Hookean solid from a Mooney–Rivlin solid) is zero or the torsion parameter τ is zero, then the solution to the boundary-value problem can be found only in terms of the axial shear deformation field; if these two parameters are not zero—even if they are small—then the latent mode of deformation is quantitatively appreciable.

In conclusion, we suggest that it is not crucial to determine the class of materials for which a given deformation field is possible. Rather, it is crucial to classify all the latent deformations associated with a given deformation field in such a way that this field is controllable for the entire class of materials. Indeed, no *real* material, even when we accept that its mechanical behaviour is purely elastic, is ever going to be described exactly by a special choice of strain-energy functions. Looking for special classes of materials for which special deformation fields are admissible may mislead us in our understanding of the nonlinear mechanical behaviour of materials.

To finish the paper on a light note, we evoke a classic wine party dilemma: which kind of corkscrew system requires the least effort to uncork a bottle? Figure 4 sketches the two working principles commonly found in commercial corkscrews. The most common type (figure 4*a*) relies on pulling only (directly or through levers) and the other type (figure 4*b*) relies on a combination of pulling and twisting. Notwithstanding the shortcomings of this paper's modelling with respect to an actual uncorking, the authors are confident that they have provided a scientific argument to those wine amateurs who favour the second type of corkscrews over the first.

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