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# Compact travelling waves in viscoelastic solids

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**Abstract** – We introduce a model for nonlinear viscoelastic solids, for which travelling shear waves with compact support are possible. Using analytical and numerical methods, we investigate the general case of this model, and an exact, kink-type travelling-wave solution is obtained as a special case result. Additionally, we derive and examine a new Burgers' type evolution equation based on the introduced constitutive equations.



**Introduction.** – A *compact wave* is a nonlinear solitary wave with a definite amplitude that exists within the confines of a compact support; outside that support, it vanishes identically. Compact waves can be used to describe patterns with a compact support and sharp fronts. While ubiquitous in Nature, compact waves are difficult to model mathematically. The motivation behind the modeling of such waves was the 1993 discovery of *compactons* by Hyman and Rosenau [1]. A compacton, in full analogy with the definition of a soliton, is a compact wave that preserves its shape and amplitude after a collision with another compact wave.

Compact waves emerge essentially from a mathematical degenerancy in the equations of motion, leading to a local loss of uniqueness, which enables the patching together of two or more different solutions with a certain degree of regularity. This point may be clarified and made rigorous in several ways; see, *e.g.*, Saccomandi [2] and Destrade *et al.* [3], who employed the classical Weierstrass criterion to generate compact waves, and also Gaeta *et al.* [4].

From a purely mathematical point of view, there exist several equations that are *factories* of compact waves, such as, *e.g.*, the family of equations $K(m,n)$ noted in [1]. On the other hand, there exist very few examples of physically based equations which are capable of generating compact waves within the framework of a rigorous theory of material behavior. For solids, three examples of such rigorous derivations have been presented thus far: i) in 1998, Dusuel *et al.* [5] showed that, in the continuum limit, the generalized $\Phi$-four, or double-well model with nonlinear coupling, can exhibit compacton-like kink solutions when the nonlinear coupling term is dominant; ii) Destrade and Saccomandi [6] used a nonlinear theory of dispersion compatible with the axiomatic foundation of simple materials; and iii) Duričković *et al.* [7] extended the theory of rods to nonlinear material laws. In the case of fluids, the emergence of compact coherent structures, mainly via asymptotic methods, has been a more investigated topic; see, *e.g.*, [8]. The reader is also referred to Saccomandi and Sgura [9], who analyzed nonlinear stacking interactions in simple models of double-stranded DNA.

Here, we show that it is possible to generate compact waves in the framework of the nonlinear theory of viscoelasticity. To this end, we consider a special viscoelastic theory of incompressible isotropic solids for which the viscous part is that of the Navier-Stokes theory, with a shear-dependent viscosity —a common assumption in non-Newtonian fluid mechanics.

In this framework, for a special class of constitutive equations, we provide a rigorous existence result, an exact (albeit implicit) travelling-wave solution, and approximations for a particular compact kink. Moreover, we derive a new generalized Burgers' equation as an asymptotic reduction of the full equation for the propagation of shear waves.

**Governing equations.** – We call $\boldsymbol{X}$ the position of a particle in the solid in $(\mathcal{B}_r)$, the reference configuration, and $\boldsymbol{x}$ the position of that particle at time $t$ in $(\mathcal{B})$, the

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current configuration. A motion of the body is the one-to-one mapping $\boldsymbol{\chi}$ such that $\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)$. The deformation gradient $\boldsymbol{F}$ and the left Cauchy-Green tensor $\boldsymbol{B}$ associated with this motion are

$$\boldsymbol{F}=\frac{\partial\boldsymbol{\chi}}{\partial\boldsymbol{X}},\quad \boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^T, \tag{1}$$

respectively, and the strain-rate tensor is defined as $\boldsymbol{D}=\frac{1}{2}(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}+\boldsymbol{F}^{-T}\dot{\boldsymbol{F}}^T)$, where the superposed dot denotes the material time derivative. An incompressible solid can undergo only isochoric motions, and this internal constraint translates mathematically as: $\det\boldsymbol{F}=1$, $\operatorname{tr}\boldsymbol{D}=0$, at all times.

We are interested in viscoelastic materials of differential type, with Cauchy stress tensor

$$\boldsymbol{T}=-p\boldsymbol{I}+2\beta_1\boldsymbol{B}-2\beta_{-1}\boldsymbol{B}^{-1}+2\hat{\nu}\boldsymbol{D}. \tag{2}$$

Here, $p$ is the indeterminate Lagrange multiplier introduced by the incompressibility constraint, $\beta_1$ and $\beta_{-1}$ are the elastic response parameters, and $\hat{\nu}$ is the shear viscosity coefficient. In all generality, $\beta_i=\beta_i(I_1,I_2)$ $(i=-1,1)$, where $I_1$, $I_2$ are the first two principal invariants of the Cauchy-Green strain: $I_1=\operatorname{tr}(\boldsymbol{B})$ and $I_2=\operatorname{tr}(\boldsymbol{B}^{-1})$. We assume that $\hat{\nu}=\hat{\nu}(\operatorname{tr}(\boldsymbol{D}^2))$, and moreover, that $\hat{\nu}>0$, *i.e.* the model is dissipative.

The momentum equations, in the absence of body forces, take the form $\operatorname{div}\boldsymbol{T}=\rho\partial\boldsymbol{v}/\partial t$, where $\rho$ is the mass density and $\boldsymbol{v}=\partial\boldsymbol{\chi}/\partial t$ is the velocity. Our aim is to investigate what happens in the *shearing motion*, specifically, $x=X+\mathfrak{f}(Z,t)$, $y=Y$, $z=Z$, where the function $\mathfrak{f}$ is as yet unknown. Straightforward computations give the components as

$$(B_{ij})=\begin{pmatrix}1+K^2 & 0 & K\\ 0 & 1 & 0\\ K & 0 & 1\end{pmatrix},$$

and

$$2\,(D_{ij})=\begin{pmatrix}0 & 0 & K_t\\ 0 & 0 & 0\\ K_t & 0 & 0\end{pmatrix},$$

where $K\equiv\mathfrak{f}_Z$ is the *amount of shear*, and the subscript denotes partial differentiation. Clearly we have now $I_1=I_2=3+K^2$ and $\operatorname{tr}(\boldsymbol{D}^2)=K_t^2/2$.

With $p_x\equiv0$, two of the three equations of motion are identically satisfied. The remaining equation is $\rho\mathfrak{f}_{tt}=\partial T_{13}/\partial Z$. Therefore the determining equation for the amount of shear $K$ becomes

$$\rho K_{tt}=\left[\hat{Q}(K^2)K+\hat{\nu}(K_t^2)K_t\right]_{ZZ}, \tag{3}$$

where $\hat{Q}\equiv2(\beta_1+\beta_{-1})$ is the *generalized shear modulus*. The mathematical theory of quasilinear equations for viscoelasticity of strain-rate type can be found in ref. [10].

In order to rewrite eq. (3) in a dimensionless form, we need a characteristic frequency $\Omega(>0)$ so that it is possible to introduce the dimensionless time $\tau=\Omega t$; we also need a characteristic length $L$ so that we can introduce the dimensionless length $\zeta=Z/L$. Usually the length $L$ is determined by the geometry of the problem (*e.g.*, the thickness of a slab wherein the wave is propagating). The characteristic frequency $\Omega$ may be introduced in several ways: via the boundary conditions, by defining the ratio $\Omega=\hat{\mu}_0/\hat{\nu}_0$, where $\hat{\mu}_0=\lim_{K^2\to0}\hat{Q}$ is the infinitesimal shear modulus and $\hat{\nu}_0=\lim_{K_t\to0}\hat{\nu}$, or by a characteristic (finite) time $t^*$ at which, for example, localization of the solution occurs. Eventually, eq. (3) becomes

$$\delta K_{\tau\tau}=[QK+\nu K_\tau]_{\zeta\zeta}, \tag{4}$$

where $\delta=\rho\Omega^2/\hat{\mu}_0L^2$, $Q=\hat{Q}/\hat{\mu}_0$, and $\nu=\Omega\hat{\nu}/\hat{\mu}_0$.

As the final step, we specialize the equation of motion (via the constitutive relations) to the case of fourth-order elasticity. In particular, we take $Q=1+\mu_1K^2$, where $\mu_1$ is a constant ($\mu_1>0$ for strain-stiffening solids and $\mu_1<0$ for strain-softening solids), and assume the simplest form of shear viscosity dependence, namely, $\nu=\nu_0+\nu_1K_\tau^2$, where $\nu_1>0$ is a constant. Note that $\mu_1=(\mu+A/2+D)/\mu$, where $\mu$, $A$, and $D$ are the second-, third-, and fourth-order constants of weakly nonlinear elasticity [11,12]. Under these assumptions, eq. (4) reduces to

$$\delta K_{\tau\tau}=\left[K+\mu_1K^3+\nu_0K_\tau+\nu_1K_\tau^3\right]_{\zeta\zeta}. \tag{5}$$

The existence and regularity of solutions of the Cauchy problem for equations of this kind have been investigated by Friedman and Necas [13]. More recently, Pucci and Saccomandi [14] considered the quasistatic limit of eq. (5) and studied the mathematical and mechanical properties of the classical creep and recovery experiments.

Before beginning our analysis, it is important to point out that eq. (5) is neither integrable nor exactly linearizable[1], as may be checked via symmetry arguments [15]. Thus, we do not expect the compact travelling waves that we will encounter in the next section to preserve their shape and amplitude after colliding. In all likelihood, these waves are not the *true* compactons of ref. [1].

**Kinks. –** In this section, we seek travelling-wave solutions (TWSs) of eq. (5) in the form of kinks; *i.e.*, continuous, bounded, monotonic waveforms that tend to constant, but unequal, limits at $\pm\infty$. It is known that *kinks* may propagate in a viscoelastic medium; Jordan and Puri [16,17] give an explicit characterization of such waves, and a detailed survey of the various qualitative properties of travelling-waves solutions in viscoelasticity may be found in [18].

We begin our search for TWSs with the following observation: since eq. (5) is invariant under the transformation

[1]In the sense that Burgers' equation *is* via the Cole-Hopf transformation.

$\zeta \mapsto -\zeta$, we need only consider, without loss of generality, right-travelling waves, *i.e.*, solutions of the specific form $K(\zeta,\tau)=g(\xi)$, where $g$ is a function of the single variable $\xi \equiv \zeta - ct$ and the positive constant $c$ denotes the wave speed. Substitution of the travelling-wave ansatz into eq. (5) results in the following nonlinear ordinary differential equation (ODE):

$$(1-\delta c^2)g'' + \mu_1(g^3)'' - \nu_0 c g''' - \nu_1 c^3[(g')^3]'' = 0, \qquad (6)$$

where primes denote differentiation with respect to $\xi$. Integrating this ODE twice, setting the first integration constant to zero, and then enforcing the usual (kink) asymptotic conditions[2] yields

$$(g')^3 + \tilde{\nu} g' = \frac{(1-\delta c^2)g + \mu_1(g^3 + g_1^2 g_2 + g_2^2 g_1)}{\nu_1 c^3}. \qquad (7)$$

Here, $\tilde{\nu} = c^{-2}\nu_0/\nu_1$ and the wave speed is given by

$$c = \delta^{-1/2}\sqrt{1+\mu_1(g_1^2 + g_1 g_2 + g_2^2)}. \qquad (8)$$

Having obtained the first-order ODE satisfied by our TWS and determined the wave speed, it is instructive to now consider the cases $\tilde{\nu}=0$ and $\tilde{\nu}>0$, separately.

*The case* $\nu_0 = 0$. Focusing on this, the simplest case first, we further simplify the analysis by taking $g_2=0$ and setting $f = g/g_1$; in particular, $c$ reduces to $c_1$, where

$$c_1 = \sqrt{(1+\mu_1 g_1^2)/\delta}, \qquad (9)$$

and eq. (7) becomes

$$(f')^3 = -\sigma f(1-f^2), \quad f \in [0,1], \qquad (10)$$

where we have set $\sigma = c_1^{-3}\mu_1/\nu_1$ for convenience. A standard stability analysis of eq. (10) reveals that $f=0,1$, the equilibrium solutions relevant to our investigation, are stable and unstable, respectively, for $\sigma>0$. Henceforth limiting our attention to only those dispersive solids that stiffen in shear (*i.e.*, those for which $\mu_1>0$), we separate variables in eq. (10) and integrate. We are thus led to consider the quadrature

$$\int \frac{\mathrm{d}f}{[f(1-f^2)]^{1/3}} = -\sigma\xi + \mathcal{K}, \qquad (11)$$

where the integration constant $\mathcal{K}$ will be chosen so that the kink is centered at $f(0)=1/2$.

Now because of the zeros at $f=0$ and $f=1$ in the denominator of the integrand, the left-hand side of eq. (11) is, in fact, a generalized integral (see [3]). Nevertheless, it can be evaluated exactly in terms of special functions. Omitting the detail, it is readily established that the exact, albeit implicit, solution is given by

$${}_2F_1\left(\tfrac{1}{3},\tfrac{1}{3};\tfrac{4}{3};f^2\right)f^{2/3} - {}_2F_1\left(\tfrac{1}{3},\tfrac{1}{3};\tfrac{4}{3};\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)^{1/3} = -\tfrac{2}{3}\sigma^{1/3}\xi, \qquad (12)$$

[2]That is, $g \to g_{1,2}$, as $\xi \to \mp\infty$, where $g_1 > g_2 \geqslant 0$ are constants.

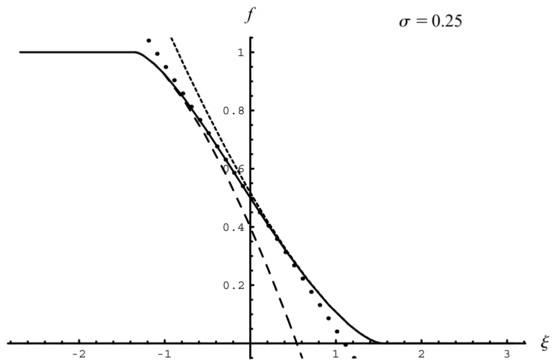


Fig. 1: $f$ *vs.* $\xi$ for $\sigma=0.25$ and $\tilde{\nu}=0$. Solid curve: eq. (12) and eq. (13). Broken curve (long dashes): eq. $(17)_1$. Dots: eq. $(17)_2$. Broken curve (short dashes): eq. $(17)_3$.

for $\xi \in (\xi_1, \xi_0)$, while outside this interval we have

$$f(\xi) = \begin{cases} 1, & \xi \leqslant \xi_1, \\ 0, & \xi \geqslant \xi_0. \end{cases} \qquad (13)$$

Here, ${}_2F_1$ denotes the Gauss hypergeometric series and the constants $\xi_1$, $\xi_0$ are defined by

$$\xi_1 = \frac{-2\pi\sqrt{3} + 3B_{1/4}\left(\frac{1}{3},\frac{2}{3}\right)}{6\sigma^{1/3}}, \quad \xi_0 = \tfrac{1}{2}\sigma^{-1/3}B_{1/4}\left(\tfrac{1}{3},\tfrac{2}{3}\right), \qquad (14)$$

where

$$B_q(a,b) \equiv \int_0^q \vartheta^{a-1}(1-\vartheta)^{b-1}\mathrm{d}\vartheta, \qquad q>0, \qquad (15)$$

denotes the *incomplete beta function*. Also, the *shock layer* thickness, $\ell$, for this kink profile has the value

$$\ell = \left[\lim_{\xi\to-\infty} f(\xi) - \lim_{\xi\to+\infty} f(\xi)\right]/|f'(0)| = 2(3\sigma)^{-1/3}. \qquad (16)$$

If we now expand the first term on the left-hand side of eq. (12) about $f=1$, $\frac{1}{2}$, 0 and then neglect the appropriate higher-order terms, the resulting expressions can be solved for $f$ in terms of $\xi$. Omitting the details, it is a relatively straightforward task, using the Heaviside unit step function, $H(\cdot)$, to construct the respective approximations

*see eq.* (17) *on the next page*

As fig. 1 makes clear, the relatively simple approximate expressions given in eq. (17) are in very good agreement with the exact kink solution within, and even somewhat outside of, their stated ranges of validity.

*Remark 1*: From eq. (16) it is evident that our kink "shocks-up," *i.e.*, the $f$ *vs.* $\xi$ profile tends to a step function, as $\sigma\to\infty$ since $\ell\to 0$ in this limit (see fig. 2). In contrast, our kink solution does *not* exhibit acceleration waves, also known as "weak discontinuities" ([19], §89), at $\xi=\xi_{1,0}$ because $f'(\xi)\in C(\mathbb{R})$; however, it should be noted that $\max_{\xi\in\mathbb{R}}|f'|\to\infty$ as $\sigma\to\infty$ (again, see fig. 2).

$$f(\xi) \simeq \begin{cases} H(\xi_1-\xi)+H(\xi-\xi_1) \\ \qquad \times\left\{1-\frac{\sqrt{2}}{27}\left[2\pi\sqrt{3}-3B_{1/4}\left(\frac{1}{3},\frac{2}{3}\right)+6\sigma^{1/3}\xi\right]^{3/2}\right\}, & \xi<-\frac{1}{2}\ell; \\ \frac{1}{2}(1-2\xi/\ell), & |\xi|\ll\frac{1}{2}\ell; \\ H(\xi_0-\xi)\left[{}_2F_1\left(\frac{1}{3},\frac{1}{3};\frac{4}{3};\frac{1}{4}\right)\left(\frac{1}{4}\right)^{1/3}-\frac{2}{3}\sigma^{1/3}\xi\right]^{3/2}, & \xi>\frac{1}{2}\ell. \end{cases} \tag{17}$$

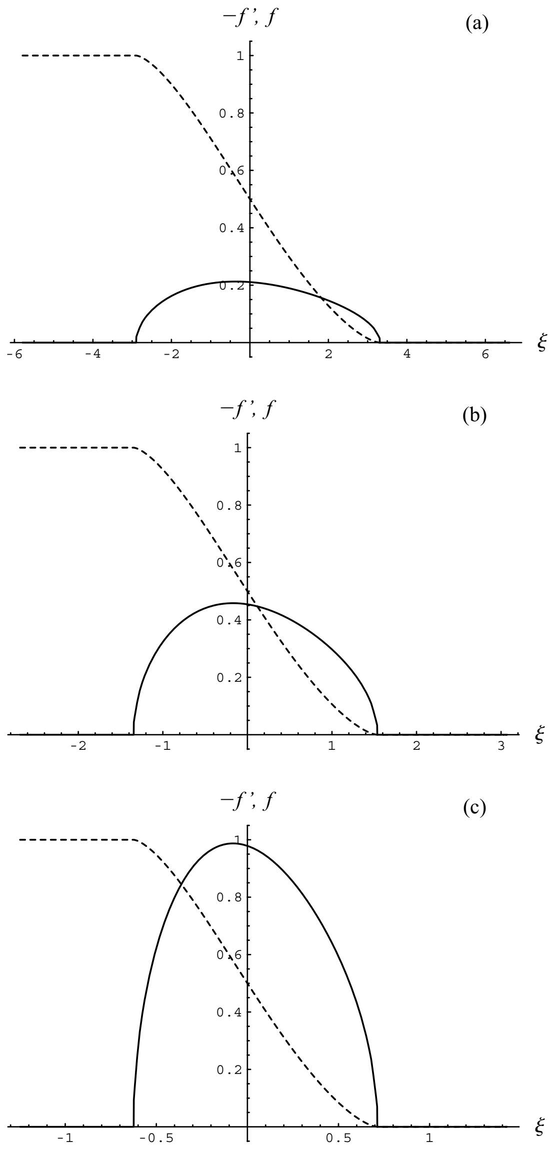


Fig. 2(a)–(c): Plotted for $\sigma=0.025, 0.25, 2.5$, respectively, and $\tilde{\nu}=0$. Solid: $-f'$ *vs.* $\xi$. Broken: $f$ *vs.* $\xi$.

*The case* $\nu_0>0$. Assuming now $\nu_0>0$, we return to eq. (7) and regard this ODE as a cubic polynomial in $g'$. Using Cardano's formula, the three roots of this cubic, each of which is a function of the single variable $g$, are readily determined. Fortunately, however, the cubic discriminant, which we denote here as $\Delta$, is always positive. Hence, only one of the roots is real-valued. Denoting this particular root by $\mathcal{G}(g)$, eq. (7) becomes

$$g'=\mathcal{G}(g)\equiv\sqrt[3]{G/2+\sqrt{\Delta}}-\frac{\tilde{\nu}}{3\sqrt[3]{G/2+\sqrt{\Delta}}}, \tag{18}$$

where $\Delta\equiv(\tilde{\nu}/3)^3+(G/2)^2$ and $G=G(g)$ denotes the right-hand side of eq. (7). We observe here that $\mathcal{G}(g)$ is strictly negative for $g\in(g_2,g_1)$, by Descartes' rule of signs, and that $g^*=\{-(g_1+g_2),g_2,g_1\}$ are the roots of both $G(g^*)=0$ and $\mathcal{G}(g^*)=0$, where $-(g_1+g_2)$ is an extraneous root in the present context. It should also be noted that the compact kink results discussed in the previous subsection are recovered with little difficulty by letting $\tilde{\nu}\to 0$ (*i.e.*, $\nu_0\to 0$).

For general values of $\tilde{\nu}$, it is possible to show the existence of a kink solution, but it does not appear possible to determine its exact analytical representation. On the other hand, because $\mathcal{G}=G/\tilde{\nu}-G^3/\tilde{\nu}^4+\cdots$, when expanded about $G=0$, it is clear that as $\tilde{\nu}\to 0$, the points where $G=0$ are associated to a $g'$ which is more and more vertical. This means that the kink *compactifies* as $\tilde{\nu}\to 0$; *i.e.*, as the nonlinear viscoelastic part in the constitutive function becomes more important with respect to the linear part, the *tails* of the kink are of less importance.

The process just described is clearly illustrated in fig. 3, where the parameter $\lambda=\tilde{\nu}/g_1^2$ has been introduced for convenience. The sequence presented therein, which was generated by solving the $g_2=0$ special case of eq. (18) numerically[3], depicts the formation of the compact kink profile shown in figs. 1 and 2(b) as $\tilde{\nu}\to 0$.

Continuing under the assumption $g_2=0$, and with $g=g_1 f$, let us now expand $\mathcal{G}=\mathcal{G}(f)$ about $f=0$. On making the additional assumption $\lambda\gg\sigma^{2/3}$ and neglecting terms $\mathcal{O}(f^4)$, eq. (18) is reduced to the following special case of Abel's equation: $f'\approx-\sigma\lambda^{-1}(f-f^3)$. Then, taking $f(0)=1/2$, the "exact" solution of this ODE is easily found, using eqs. (12)–(18) of ref. [16], to be

$$f(\xi)\approx\frac{1}{\sqrt{1+3\exp(2\sigma\xi/\lambda)}}\quad(\lambda\gg\sigma^{2/3}), \tag{19}$$

with a shock layer thickness of $L_{\mathrm{Abel}}=\frac{8}{3}\lambda/\sigma$ that can never go to zero. Here, we observe that the graph of eq. (19) is very similar, qualitatively speaking, to the one shown in fig. 3(a).

*Remark 2*: From eq. (19) we find that, for sufficiently large values of $\lambda$, and a zero limit as $\xi\to+\infty$,

[3]Specifically, the NDSolve[ ] routine, which is provided in the software package MATHEMATICA (version 5.2), was employed here.

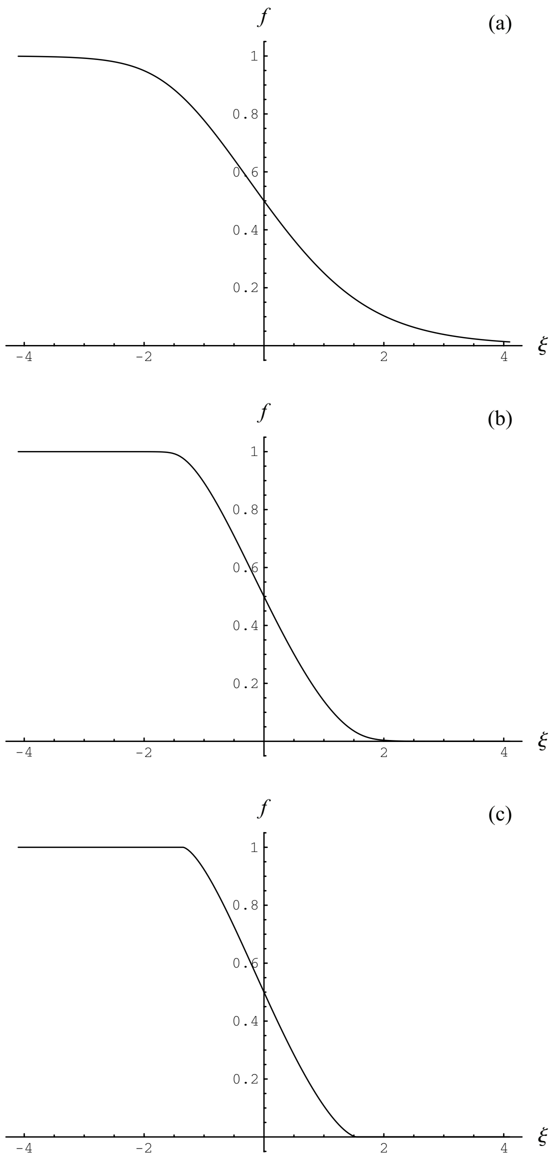


Fig. 3(a)–(c): $f$ *vs.* $\xi$ for $\lambda = 0.25, 0.05, 0.001$, respectively, and $\sigma = 0.25$. (Here, as before, $f = g/g_1$.)

the viscoelastic model considered in the present letter behaves very much like the cubically perturbed Kelvin-Voigt model; see, *e.g.*, ref. [16] and those therein.

**A new Burgers' equation. –** Let us return to eq. (5) and introduce the new independent variables $\theta = \tau - a_0\zeta$ and $s = \epsilon\zeta$, where $a_0 = \delta^{-1/2}$ and $\epsilon$ is a small parameter. If we now set $K(\zeta,\tau) = \epsilon^{1/2}\kappa(s,\theta)$ for some function $\kappa$, then eq. (5) can be approximated *via* a generalized form of the modified Burgers' equation (MBE)[4], namely,

$$\kappa_s = \bar{\mu}_1(\kappa^3)_\theta + \bar{\nu}_0\kappa_{\theta\theta} + \bar{\nu}_1(\kappa_\theta^3)_\theta, \tag{20}$$

with the latter reducing to the former in the limit $\bar{\nu}_1 \to 0$. Here, we have set $\bar{\mu}_1 = \mu_1 a_0/2$, $\bar{\nu}_0 = \nu_0 a_0/2$, and $\bar{\nu}_1 = \nu_1 a_0/2$; and we have assumed $\nu_0 = \mathcal{O}(\epsilon)$. Additionally, we note that when $\nu_0 = 0$, the structure of the travelling-wave solutions of eqs. (5) and (20) is exactly the same.

Moreover, because eq. (20) is an evolution equation, which is simpler than the wave equation given in eq. (5), other reductions to ordinary differential equations are easily found. For example, when $\bar{\nu}_0 = 0$, eq. (20) admits solutions in the separable form $\kappa(s,\theta) = \phi(s)\psi(\theta)$, say. If $\bar{\nu}_0 = 0$ and $\bar{\mu}_1 = 0$, then eqn. (20) is a *degenerate* diffusion equation and the separable solutions are quite simple, with an interesting structure. Indeed, we have

$$\kappa(s,\theta) = \frac{\gamma_1(\theta-\gamma_2)^2}{\sqrt{2\gamma(\gamma_3 - s)}}. \tag{21}$$

Here, $\gamma$ is the separation constant and $\gamma_1$, $\gamma_2$, $\gamma_3$ are integration constants such that $24\gamma_1^2\bar{\nu}_1 - \gamma = 0$, and therefore $\gamma, \gamma_3 > 0$. Clearly, these solutions have a sharp front $s = \gamma_2$; and they blow-up in space for $s = \gamma_3$.

It is interesting to consider the case of harmonic excitation and the corresponding third-harmonic order generation. Given the initial condition $\kappa(0,\theta) = \kappa_0\cos(\omega\theta)$, where $\kappa_0$ is a positive constant, we assume the solution is given by the sum of the fundamental and third-harmonic components; *i.e.*, $\kappa = \kappa_1 + \kappa_3$, where

$$\begin{aligned}\kappa_1(s,\theta) &= \mathrm{Re}\{\hat{\kappa}_1(s)\exp(\mathrm{i}\omega\theta)\} = \tfrac{1}{2}\hat{\kappa}_1(s)\exp(\mathrm{i}\omega\theta) + \text{c.c.},\\ \kappa_3(s,\theta) &= \mathrm{Re}\{\hat{\kappa}_3(s)\exp(3\mathrm{i}\omega\theta)\} = \tfrac{1}{2}\hat{\kappa}_3(s)\exp(3\mathrm{i}\omega\theta) + \text{c.c.}.\end{aligned}$$

Here, "c.c." denotes the complex conjugate of the preceding term and we require that $|\kappa_3| \ll |\kappa_1|$. By successive approximation, we have

$$\frac{\mathrm{d}\hat{\kappa}_1}{\mathrm{d}s} + \alpha\hat{\kappa}_1 = 0, \tag{22}$$

where $\alpha = \omega^2\bar{\nu}_0$, and

$$\frac{\mathrm{d}\hat{\kappa}_3}{\mathrm{d}s} + 9\alpha\hat{\kappa}_3 = -\tfrac{3}{4}\omega\hat{\kappa}_1^3(\mathrm{i}\bar{\mu}_1 - \omega^3\bar{\nu}_1), \tag{23}$$

where the latter is solved subject to $\kappa_3(0,\theta) = 0$.

Solving these ODEs in sequence using one of the many standard methods, we find, in turn, that

$$\begin{aligned}\kappa_1 &= \kappa_0\mathrm{e}^{-\alpha s}\cos(\omega\theta),\\ \kappa_3 &= \frac{\kappa_0^3}{8\alpha}(\mathrm{e}^{-3\alpha s} - \mathrm{e}^{-9\alpha s})\\ &\quad\times[\bar{\mu}_1\omega\sin(3\omega\theta) + \bar{\nu}_1\omega^4\cos(3\omega\theta)].\end{aligned} \tag{24}$$

These solutions reveal that the experimental measurement of the third harmonic in soft solids with shear-dependent viscosity gives direct access to the nonlinear shear wave elastic parameter $\bar{\mu}_1$ and nonlinear dissipation parameter $\bar{\nu}_1$. The former is measured in the low-frequency regime, whilst the latter becomes dominant in the high-frequency regime.

*Remark 3*: If we let $\bar{\nu}_1 \to 0$, and make the associations $\bar{\mu}_1 \mapsto -\frac{1}{3}c^{-3}\beta$ and $\bar{\nu}_0 \mapsto \delta$, then $\kappa_1$ and $\kappa_3$, respectively,

[4] So referred to as by Lee-Bapty and Crighton in 1987; see refs. [12,16] and those therein.

reduce to $v_1$ and $v_3$, which correspond to the MBE, given in [12], eq. (42).

*Remark 4*: In attempting to satisfy $|\kappa_3| \ll |\kappa_1|$, it is helpful to know that

$$\max_{\theta>0} |\kappa_3| \leqslant \frac{\kappa_0^3}{12\alpha\sqrt{3}} \left\{\bar{\nu}_1\omega^4 + |\bar{\mu}_1|\omega\right\}, \qquad (25)$$

where

$$\max_{\theta>0} |\kappa_3| = |\kappa_3|\Big|_{\theta=\theta^*} = \frac{\kappa_0^3|\bar{\nu}_1\omega^4\cos(3\omega s) + \bar{\mu}_1\omega\sin(3\omega s)|}{12\alpha\sqrt{3}}, \qquad (26)$$

and where $\theta^* = (6\alpha)^{-1}\ln(3)$.

*Remark 5*: The inequality $\max_{\theta>0} |\kappa_3| < \sup_{\theta>0} |\kappa_1| (= \kappa_0)$ is satisfied $\forall\, \omega \in (0, \omega^+)$, where $\omega^+$ is the only positive root of $\bar{\nu}_1\omega^4 + |\bar{\mu}_1|\omega - 12\kappa_0^{-2}\alpha\sqrt{3} = 0$.

**Concluding remarks.** – We have established an important new constitutive framework that originates field equations admitting compact kinks. This constitutive framework, which provides one more example of how compactification may arise in the modeling of real-world wave phenomena, is relevant because it is a natural model for nonlinear viscoelasticity; indeed, the mechanism of compactification described here is directly linked to the nonlinear viscosity term. We have also derived from this model a new Burgers' type evolution equation, of which the MBE is a limiting case, that actually maintains the compactification features of the general case. And by adopting the usual successive approximation method used to study harmonic generation in nonlinear acoustics, we have shown that an important difference between our model and the classical one occurs in the higher harmonics.

Lastly, it should be noted that the mechanism which generates the compact waves described in this letter is completely different from those presented in [3,7,9], where the compactification arose from an interplay between nonlinearity and dispersion, and in [4], where compactification was made possible by considering non-smooth potentials.

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