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SURFACE STABILITY ANALYSIS OF A PREDEFORMED BELL-CONSTRAINED HALF-SPACE

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Abstract.

Near-the-surface instability is examined for a finitely and homogeneously deformed hyperelastic semi-infinite body subject to the Bell constraint, by using surface (Rayleigh) waves. These inhomogeneous plane waves propagate in the direction of a principal axis of the finite homogeneous static deformation, and decay exponentially away from the free flat surface of the half-space, in the direction of another principal axis. The exact secular equation, giving the speed of propagation v*, is found. Then by letting* v *tend to zero, the 'bifurcation criterion' or 'stability equation' is established; this equation delimits a surface in the stretch ratios space which separates a region where the deformed half-space is always stable with respect to perturbations from a region where the half-space might be unstable. Finally two specific examples of materials are treated: 'Bell's empirical model' and 'Bell simple hyperelastic material', and it is seen that in each case, the stability equation is universal to the whole subclass considered.*

1 INTRODUCTION

According to the Introduction of a recent textbook by Guz,[1] the three-dimensional theory of deformable bodies stability can be traced back to the early works of Biot[2,3] in the 1930's, at least as far as incremental deformations of finitely prestressed bodies are concerned. In a series of articles, most of which are summarized in his book,[4] Biot analyzed both the internal and the near-the-surface instabilities of homogeneous as well as heterogeneous prestressed elastic and viscoelastic materials. He pointed out that the study of surface instability is formally analogous to the study of surface (Rayleigh) waves.

Surface waves are indeed a valuable tool when the stability of an elastic half-space $x_2 \geq 0$ (say) comes under study. This is so because they may be decomposed into a combination of inhomogeneous plane waves, propagating with speed v on the flat surface $x_2 = 0$, in a direction x_1 (say), and with attenuation in the x_2 direction, so that they are perturbations of the form

$$\epsilon \Re\{\mathbf{A} e^{ik(x_1 + \xi x_2 - vt)}\}. \tag{1}$$

Here, ϵ is a real parameter, 'small' in the sense that terms of order higher than ϵ^1 may be neglected; $\mathbf{A}$ is a complex vector, describing the polarization of the wave; k is the real wave number; and ξ is a complex scalar such that $\Im(\xi) > 0$, in order for the amplitude to vanish away from the surface $x_2 = 0$. Now consider a semi-infinite body, made of some hyperelastic material, maintained in an equilibrium state of finite pure homogeneous deformation under uniform compressive (or tensile) loads P_1 and P_3. Let x_1, x_2, x_3, be the three principal axes of the deformation, $x_2 = 0$ be the flat surface delimiting the upper end of the body, and let a wave such as (1) propagate near this surface. Once the equations of motion are solved and the boundary conditions are satisfied, an equation – the *secular equation* – is obtained for v^2. The parameters in this equation depend on the material constants characteristic of the elastic body and on the principal stretch ratios of the primary static deformation. For a given configuration, a unique positive root of the secular equation is usually found for v^2, and the configuration is said to be stable with respect to a perturbation of the form (1), or using Fourier analysis, with respect to any dynamical perturbation. However, it may happen that for certain stretch ratios, the secular equation admits $v^2 = 0$ as a root. This phenomenon corresponds to the vanishing of the apparent surface rigidity[4] and is generally associated with the appearance of 'ripples' or 'wrinkles' on the free surface of the half-space. Furthermore, there might exist some configurations where the secular equation admits only negative roots $v^2 < 0$, leading to a perturbation (1) increasing exponentially with time and rendering the linear approximation of small perturbations superimposed on large invalid. The secular equation written at $v^2 = 0$ is usually called the *stability equation* or the *bifurcation criterion*. It defines a surface in the stretch ratios space which, in the linearized theory, separates stable configurations of the half-space from unstable configurations.

Using this background, the present article establishes the secular equation for surface waves on a deformed Bell-constrained half-space in order to study its stability. The Bell constraint was obtained experimentally by James F. Bell[5] for the behavior of certain annealed metals and has since been extensively studied in a theoretical manner within the context of hyperelasticity (see Beatty[6] and the references therein). Note that the secular equation for this problem was recently found[7] using the method of first integrals proposed by Mozhaev.[8] This method is rapid and elegant but presents a minor inconvenient because the secular equation is obtained in a polynomial form (a cubic in v^2) which corresponds to the rationalization of the *exact* secular equation and the introduction of spurious roots.[9,10] Consequently, there is a need to obtain the exact secular equation, from which the exact bifurcation criterion may be deduced. This procedure is carried on in Section 3, using previously established results recalled in Section 2. Then in Section 4, the analysis is specialized to two specific classes of Bell-constrained materials. First, for 'Bell's empirical model', whose strain energy function depends upon only one material parameter, the bifurcation criterion found by Beatty and Pan[11] is quickly recovered; second, for a 'simple hyperelastic Bell material', whose strain energy function depends upon two material parameters, the bifurcation criterion is found to be a very simple relationship between the stretch ratios. In both cases, the bifurcation criterion is universal to the whole subclass considered, as it does not depend upon any material constant. In the latter case, the influence of the prestrain on the speed of the Rayleigh waves and on the stability of the half-space is discussed. Finally, comparisons are made with the stability of half-spaces subject to the constraint of incompressibility (which coincides with the Bell constraint in the isotropic limit[12]), specifically with neo-Hookean and Mooney-Rivlin materials, which model the behavior of rubber.

2 STABILITY EQUATIONS

2.1 Basic deformation of a Bell half-space

Let $(O, x_1, x_2, x_3) \equiv (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a Cartesian rectangular coordinate system. Let the half-space $x_2 \geq 0$ be occupied by a hyperelastic Bell-constrained material, with strain energy density Σ. This material is subject to the internal constraint that for any deformation,[12,13]

$$i_1 \equiv \text{tr } \mathbf{V} = 3, \tag{2}$$

at all times, where $\mathbf{V}$ is the left stretch tensor. Hence, for isotropic Bell materials, Σ depends only upon i_2 and i_3, the respective second and third invariants of $\mathbf{V}$. So, $\Sigma = \Sigma(i_2, i_3)$, where

$$i_2 = [(\text{tr } \mathbf{V})^2 - \text{tr } (\mathbf{V}^2)]/2, \; i_3 = \det \mathbf{V}, \tag{3}$$

and the constitutive equation giving the Cauchy stress tensor $\mathbf{T}$ is[12]

$$\mathbf{T} = p\mathbf{V} + \omega_0 \mathbf{1} + \omega_2 \mathbf{V}^2, \tag{4}$$

where p is an arbitrary scalar, to be found from the equations of motion and the boundary conditions, and the material response functions ω_0 and ω_2 are defined by

$$\omega_0 = \frac{\partial \Sigma}{\partial i_3}, \quad \omega_2 = -\frac{1}{i_3}\frac{\partial \Sigma}{\partial i_2}, \tag{5}$$

and should verify the Beatty–Hayes A-inequalities[12]

$$\omega_0(i_2, i_3) \leq 0, \quad \omega_2(i_2, i_3) > 0. \tag{6}$$

In the case where the material is maintained in a state of finite pure homogeneous static deformation, with principal stretch ratios $\lambda_1, \lambda_2, \lambda_3$ along the x_1, x_2, x_3 axes, the Cauchy stress tensor is the constant tensor $\mathbf{T}_o$ given by

$$\mathbf{T_o} = (p_o\lambda_1 + \omega_0 + \lambda_1^2\omega_2)\mathbf{i}\otimes\mathbf{i} + (p_o\lambda_2 + \omega_0 + \lambda_2^2\omega_2)\mathbf{j}\otimes\mathbf{j} + (p_o\lambda_3 + \omega_0 + \lambda_3^2\omega_2)\mathbf{k}\otimes\mathbf{k}. \tag{7}$$

Here ω_0 and ω_2 are evaluated at i_2, i_3 given by

$$i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{8}$$

Of course,

$$\lambda_1 + \lambda_2 + \lambda_3 = 3, \tag{9}$$

in order to satisfy (2). It is assumed that the boundary $x_2 = 0$ is free of tractions so that $T_{o22} = 0$; and that the compressive loads P_1 and P_3 are applied at $x_1 = \infty$ and $x_3 = \infty$ to maintain the deformation so that $P_1 = -T_{o11}$ and $P_3 = -T_{o33}$. Hence,

$$p_o = -(\omega_0 + \lambda_2^2\omega_2)/\lambda_2, \quad P_\gamma = (\lambda_2 - \lambda_\gamma)(-\omega_0 + \lambda_\gamma\lambda_2\omega_2)/\lambda_2, \quad (\gamma = 1, 3). \tag{10}$$

2.2 Incremental near-the-surface motions

Beatty and Hayes[14] wrote the general equations for small-amplitude motions in a Bell-constrained material maintained in a state of finite pure homogeneous deformation (as described in the previous subsection). These equations were then specialized by this author to surface (Rayleigh) waves. For a wave of the form (1), written as $\epsilon\Re\{\mathbf{U}(x_2)e^{ik(x_1 - vt)}\}$, where $\mathbf{U}$ is an unknown function of x_2, it was proved that the equations of motion and the boundary conditions relative to surface waves could be written in a quite simple manner in terms of the *incremental tractions* σ_{12}^*, σ_{22}^*, acting upon the planes $x_2 =$ const. Explicitly, and introducing the scalars functions $t_1(x_2)$ and $t_2(x_2)$, defined by

$$\sigma_{21}^*(x_1, x_2, t) = t_1(x_2)e^{ik(x_1 - vt)}, \quad \sigma_{22}^*(x_1, x_2, t) = t_2(x_2)e^{ik(x_1 - vt)}, \tag{11}$$

it was found that the equations of motion are

$$\begin{aligned} &[\mu(\lambda_1^2 - \lambda_2^2) - X]t_1'' + i\beta_{12}t_2' - \frac{1}{\mu\lambda_2^2}(\lambda_1\lambda_2^{-1}C - X)(\mu\lambda_1^2 - X)t_1 = 0, \\ &(\lambda_1\lambda_2^{-1}C - X)t_2'' + i\beta_{12}t_1' - \lambda_1^2\lambda_2^{-2}[\mu(\lambda_1^2 - \lambda_2^2) - X]t_2 = 0, \end{aligned} \tag{12}$$

and the boundary conditions are simply

$$t_1(0) = t_2(0) = t_1(\infty) = t_2(\infty) = 0. \tag{13}$$

Here, $X = \rho v^2$ where ρ is the mass density of the material, and

$$\begin{aligned}
&\mu = \frac{-\omega_0 + \lambda_1\lambda_2\omega_2}{\lambda_2(\lambda_1+\lambda_2)},\\
&C_{\alpha\beta} = 2\lambda_\alpha^2\delta_{\alpha\beta}\omega_2 - \lambda_\beta^2(\omega_{02} + \lambda_\alpha^2\omega_{22}) + \lambda_1\lambda_2\lambda_3(\omega_{03} + \lambda_\alpha^2\omega_{23}),\\
&C = \lambda_1^{-1}\lambda_2 C_{11} + \lambda_1\lambda_2^{-1}C_{22} - C_{12} - C_{21} - 2\omega_0 - (\lambda_1^2+\lambda_2^2)\omega_2,\\
&\beta_{12} = \lambda_1\lambda_2^{-1}[\mu(\lambda_1^2-\lambda_2^2) + C] - (1+\lambda_1\lambda_2^{-1})X,
\end{aligned} \tag{14}$$

where the derivatives $\omega_{0\Gamma}$, $\omega_{2\Gamma}$ $(\Gamma = 2, 3)$ of the material response functions ω_0, ω_2 are taken with respect to i_Γ and evaluated at i_2, i_3 given by (8).

3 EXACT SECULAR EQUATION AND BIFURCATION CRITERION

Now a law of exponential decay is chosen for $\mathbf{t}(\mathbf{x_2}) = [t_1(x_2), t_2(x_2)]^{\mathrm{T}}$,

$$\mathbf{t}(x_2) = e^{ik\xi x_2}\mathbf{T}, \quad \text{with} \quad \Im(\xi) > 0, \tag{15}$$

where $\mathbf{T}$ is a constant vector. Then the incremental equations of motion (12) are

$$\begin{bmatrix} \alpha_{11}\xi^2 + \gamma_{11} & \beta_{12}\xi \\ \beta_{12}\xi & \alpha_{22}\xi^2 + \gamma_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{16}$$

where β_{12} is defined in $(14)_4$ and

$$\begin{aligned}
&\alpha_{11} = [\mu(\lambda_1^2-\lambda_2^2) - X], \quad \alpha_{22} = (\lambda_1\lambda_2^{-1}C - X),\\
&\gamma_{11} = \frac{1}{\mu\lambda_2^2}(\lambda_1\lambda_2^{-1}C - X)(\mu\lambda_1^2 - X), \quad \gamma_{22} = \lambda_1^2\lambda_2^{-2}[\mu(\lambda_1^2-\lambda_2^2) - X].
\end{aligned} \tag{17}$$

Hence, nontrivial solutions exist when ξ is root of the biquadratic

$$\xi^4 - S\xi^2 + P = 0, \tag{18}$$

where

$$S = \frac{\beta_{12}^2 - \alpha_{11}\gamma_{22} - \alpha_{22}\gamma_{11}}{\alpha_{11}\alpha_{22}}, \quad P = \frac{\gamma_{11}\gamma_{22}}{\alpha_{11}\alpha_{22}} = \frac{(\mu\lambda_1^2 - X)\lambda_1^2}{\mu\lambda_2^4} > 0. \tag{19}$$

The quantity P is positive because, owing to the A-inequalities (6), μ defined in $(14)_1$ is positive, and because in the subsonic range, a surface wave propagates a speed which is lower than that of any body wave, and in particular, lower than the pure shear wave propagating along the x_1 direction $(X < \mu\lambda_1^2)$.

Now we examine in turn different situations that may arise in the resolution of the bi-quadratic (18): **(a)** $S^2 - 4P > 0$ and $S > 0$; **(b)** $S^2 - 4P > 0$ and $S < 0$; **(c)** $S^2 - 4P < 0$ and any real value for S (we exclude the particular case of a repeated root, because it does not correspond to a surface wave[15]). Case **(a)** must be ruled out because it leads to real values for all the roots and no exponential decay for the wave as $x_2 \to \infty$. In Case **(b)**, the roots are all purely imaginary, and $\xi_1 = i\sqrt{(-S + \sqrt{S^2 - 4P})/2}$ and $\xi_2 = i\sqrt{(-S - \sqrt{S^2 - 4P})/2}$ have positive imaginary parts. In Case **(c)**, the roots with positive imaginary parts are $\xi_1 = a + ib$, $\xi_2 = -a + ib$, where $a = \sqrt{(S + 2\sqrt{P})/4}$ and $b = \sqrt{(-S + 2\sqrt{P})/4}$. We conclude that in both acceptable cases **(b)** and **(c)**, we have

$$\xi_1 \xi_2 = -\sqrt{P}. \tag{20}$$

Note that the different cases **(a)**, **(b)**, **(c)** written at $X = \rho v^2 = 0$ are related to the examination of the internal (or material) stability[4] of the material. These points were treated by Pan and Beatty[16] for Bell materials and are not considered here, as this paper is concerned only with near-the-surface stability.

Now let $\mathbf{T}^{(r)}$ be a vector satisfying (16) when $\xi = \xi_r$, $(r = 1, 2)$. It may be written as

$$\mathbf{T}^{(r)} = \begin{bmatrix} 1 \\ q_r \end{bmatrix}, \quad \text{where} \quad q_r = -\frac{\alpha_{11}\xi_r^2 + \gamma_{11}}{\beta_{12}\xi_r}, \quad (r = 1, 2). \tag{21}$$

Then the tractions $\mathbf{t}$ defined in (11) are a combination of $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ for some constants A and B, $\mathbf{t}(x_2) = Ae^{ik\xi_1 x_2}\mathbf{T}^{(1)} + Be^{ik\xi_2 x_2}\mathbf{T}^{(2)}$, and they must satisfy the boundary conditions $(13)_{1,2}$, that is $\mathbf{t}(0) = \mathbf{0}$, or

$$A + B = 0, \quad q_1 A + q_2 B = 0. \tag{22}$$

This linear homogeneous system of two equations for the two unknowns A and B has non trivial solutions when $q_2 - q_1 = 0$, that is when the following *exact secular equation* is verified, $\alpha_{11}\xi_1\xi_2 - \gamma_{11} = 0$, or using $(17)_{1,3}$, (20), and $(19)_2$,

$$[\mu(\lambda_1^2 - \lambda_2^2) - X]\sqrt{\mu\lambda_1^2} + (\lambda_1\lambda_2^{-1}C - X)\sqrt{\mu\lambda_1^2 - X} = 0. \tag{23}$$

Note that in the process of obtaining the secular equation, the spurious roots corresponding to $\xi_2 - \xi_1 = 0$ were dropped. Note also that by a squaring process, the cubic secular equation,[7] which has spurious roots, may be deduced from (23).

Now we may deduce the *bifurcation criterion* directly from (23), by letting X tend to zero, as

$$\mu(\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2^{-1}C = 0. \tag{24}$$

When for certain stretch ratios λ_1, λ_2, λ_3, this criterion is satisfied, the predeformed semi-infinite Bell medium looses its near-the-surface stability. This equation defines a surface in the space of the stretch ratios which separates a region where the homogeneous deformations of the Bell half-space are always stable (where $\rho v^2 > 0$) from a region where they might be unstable (where $\rho v^2 < 0$, indicating temporal growth, at least in the linearized theory). Of course, it must be kept in mind that the stretch ratios must always satisfy the Bell constraint (9).

4 TWO ILLUSTRATIVE EXAMPLES

4.1 Bell's empirical model

For *Bell's empirical model materials*,[5] the strain energy function Σ is given by

$$\Sigma = \tfrac{2}{3}\beta_0[2(3 - i_2)]^{3/4}, \tag{25}$$

where β_0 is a positive constant, and the material response functions ω_0 and ω_2 provided by (5) are

$$\omega_0 = 0, \quad \omega_2 = \frac{1}{i_3}\beta_0[2(3 - i_2)]^{-1/4}. \tag{26}$$

In that context,

$$\mu = \frac{\lambda_1\omega_2}{\lambda_1 + \lambda_2}, \quad C = [2 - \frac{(\lambda_1 - \lambda_2)^2}{4(3 - i_2)}]\lambda_1\lambda_2\omega_2. \tag{27}$$

Then the bifurcation criterion (24) may be arranged as

$$3 - \frac{(\lambda_1 - \lambda_2)^2}{4(3 - i_2)} - \lambda_1^{-1}\lambda_2 = 0, \tag{28}$$

which is precisely the stability equation (7.10) established by Beatty and Pan,[11] using the Euler stability criterion. It is independent of the material constant β_0. These authors also conducted a detailed analysis of the regions delimited by the bifurcation criterion.

4.2 Simple hyperelastic Bell materials

For *simple hyperelastic Bell materials*,[12] the strain energy function Σ is given by

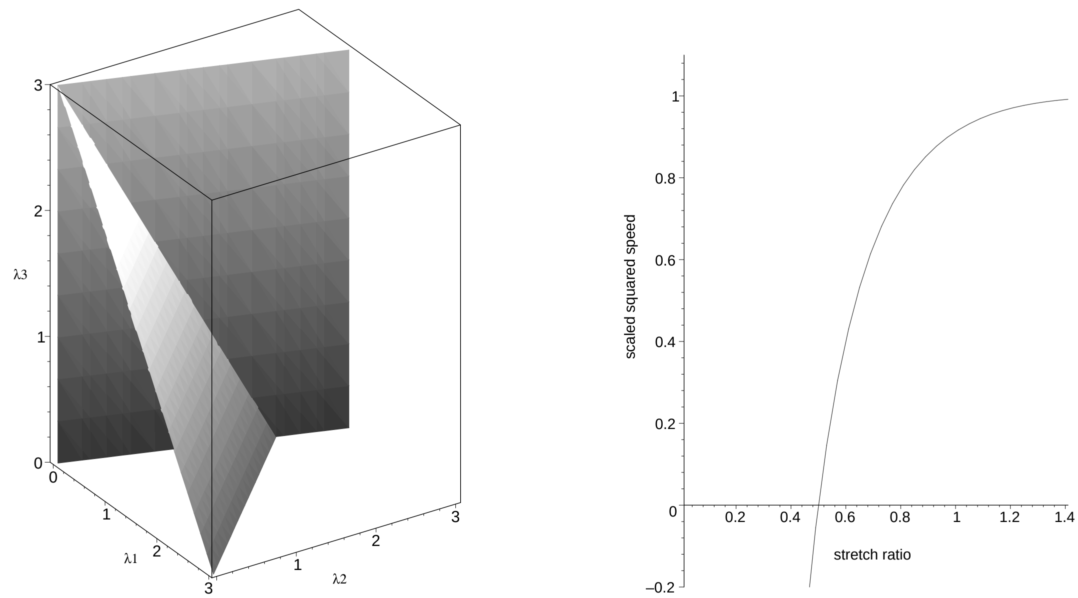
$$\Sigma = \alpha(3 - i_2) + \beta(1 - i_3), \tag{29}$$

where α and β are positive constants, and the material response functions ω_0 and ω_2 provided by (5) are now

$$\omega_0 = -\beta, \quad \omega_2 = \alpha/i_3. \tag{30}$$

In that context,

$$\mu = \frac{\beta + \alpha\lambda_3^{-1}}{\lambda_2(\lambda_1 + \lambda_2)}, \quad C = 2(\beta + \alpha\lambda_3^{-1}). \tag{31}$$



(a) Region of stability (b) Surface wave speed

Figure 1: Near-the-surface stability for simple hyperelastic Bell materials.

Then the bifurcation criterion (24) simplifies considerably to

$$3\lambda_1 - \lambda_2 = 0, \tag{32}$$

which is a particularly simple linear relationship between the stretch ratios λ_1 and λ_2, universal to the whole class of simple hyperelastic Bell materials. This equation delimits a plane in the stretch ratios space $(\lambda_1, \lambda_2, \lambda_3)$, which cuts the constraint plane (9) along the straight segment going from the point (0,0,3) to the point $(\frac{3}{4}, \frac{9}{4}, 0)$. Moreover, it will become apparent in the foregoing analysis that the region which is stable with respect to perturbations (where $X = \rho v^2 > 0$) is: $3\lambda_1 - \lambda_2 > 0$. In Figure 1(a), the plane (32) cuts the triangle of the possible values for the stretch ratios (9) into two parts, of which the visible one is the region of stability of any simple hyperelastic Bell material.

Regarding surface waves, a change of variable suggested by Dowaikh and Ogden[9] may be applied and the secular equation (23) may be written as a polynomial in η, defined by

$$\eta = \Big[1 - \frac{X}{\mu\lambda_1^2}\Big]^{\frac{1}{2}} = \Big[1 - \frac{(\lambda_1+\lambda_2)\lambda_2}{\lambda_1^2}\frac{\rho v^2}{\beta + \alpha\lambda_3^{-1}}\Big]^{\frac{1}{2}}, \tag{33}$$

as

$$f(\eta) \equiv \eta^3 + \eta^2 + (1 + 2\lambda_1^{-1}\lambda_2)\eta - \lambda_1^{-2}\lambda_2^2 = 0. \tag{34}$$

Clearly, at $\eta = 0$ (corresponding to a transverse bulk wave), we have $f(0) = -\lambda_1^{-2}\lambda_2^2 < 0$, while at $\eta = 1$ (corresponding to $v = 0$), we have $f(1) = (3\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)\lambda_1^{-2}$. So, because f is a monotone increasing function of η, the existence of a positive root for ρv^2 in the subsonic interval $[0, \mu\lambda_1^2]$ is equivalent to the condition: $3\lambda_1 - \lambda_2 > 0$ being satisfied.

In Figure 1(b), the influence of the prestrain upon the speed of the surface wave is made apparent in the case of plane strain ($\lambda_3 = 1$). On the horizontal axis, λ_1 is increased from a compressive value ($\lambda_1 < 1$) to a tensile value ($\lambda_1 > 1$). The coordinate on the vertical axis is the squared surface wave speed, scaled with respect to the transverse bulk wave speed, that is $\rho v^2/(\mu\lambda_1^2)$. At $\lambda_1 = 1$, the half-space is isotropic ($\lambda_1 = \lambda_2 = \lambda_3 = 1$) and the scaled squared speed is equal to 0.9126, the value found by Lord Rayleigh[17] in the incompressible isotropic case (Beatty and Hayes[12] have proved that the constraints of incompressibility and of Bell are equivalent for infinitesimal motions). A compressive load $P_1 > 0$, $\lambda_1 < 1$, will decrease the speed of propagation for the surface wave, until the critical stretch of $(\lambda_1)_{\text{cr}} = 0.5$ (see next subsection) where the surface looses stability. Conversely, a tensile load $P_1 < 0$, $\lambda_1 > 1$, will increase the speed of propagation for the surface wave, with the speed of the transverse bulk wave as an upper bound.

4.3 Comparisons with incompressible rubber

The stability of a deformed half-space made of incompressible rubber was first studied by Biot. He used the neo-Hookean model but noted[4] that his results were also valid for the Mooney-Rivlin model. He obtained the bifurcation criterion, showed that it was universal to both classes of materials, and computed the value of the critical stretch $(\lambda_1)_{\text{cr}}$ at which the rubber half-space looses stability under compressive loads, first in the case of plane strain, then in the case of biaxial strain. Both cases involved the numerical resolution of a cubic. Now we show that for Bell's empirical models and for simple hyperelastic Bell materials, the critical stretch can be found explicitly.

First we consider that the half-space made of Bell-constrained material is deformed in such a way that there is no extension in the x_3-direction ($\lambda_3 = 1$). This is possible when the compressive load $P_3 = (\lambda_2 - 1)(-\omega_0 + \lambda_2\lambda_3\omega_2)/\lambda_2$ is applied at infinity. Then we have $\lambda_2 = 2 - \lambda_1$ by (9). For *Bell's empirical model*, the bifurcation criterion (28) reduces to

$$3\lambda_1 - 2 = 0, \quad \text{so that} \quad (\lambda_1)_{\text{cr}} = \tfrac{2}{3}. \tag{35}$$

For *simple hyperelastic Bell materials*, the bifurcation criterion (32) reduces to

$$4\lambda_1 - 2 = 0, \quad \text{so that} \quad (\lambda_1)_{\text{cr}} = \tfrac{1}{2}. \tag{36}$$

Then we consider that the half-space made of Bell-constrained material is allowed to expand freely in the x_3-direction, so that $P_3 = 0$. Then we have $\lambda_2 = \lambda_3 = (3 - \lambda_1)/2$ by $(10)_3$ and (9). For *Bell's empirical model*, the bifurcation criterion (28) reduces to

$$11\lambda_1 - 6 = 0, \quad \text{so that} \quad (\lambda_1)_{\text{cr}} = \tfrac{6}{11}. \tag{37}$$

For *simple hyperelastic Bell materials*, the bifurcation criterion (32) reduces to

$$7\lambda_1 - 3 = 0, \quad \text{so that} \quad (\lambda_1)_{\text{cr}} = \tfrac{3}{7}. \tag{38}$$

In Table 1, the numerical values for the critical stretches are given for the classes of Bell's empirical model (2nd column), of neo-Hookean and Mooney-Rivlin incompressible materials[4] (3rd column), and of simple hyperelastic Bell materials (4th column), in the cases of plane strain (2nd row) and of biaxial strain (3rd row). It appears that rubber can be compressed more than Bell's empirical model but less than simple hyperelastic Bell materials, before it looses its near-the-surface stability.

Table 1: Critical stretch ratios for surface instability

	Bell empirical	rubber	simple Bell
$\lambda_3 = 1$	0.667	0.544	0.500
$\lambda_2 = \lambda_3$	0.545	0.444	0.429

REFERENCES

[1] A.N. Guz, *Fundamentals of the three-dimensional theory of stability of deformable bodies*, Springer-Verlag, Berlin (1999).
[2] M.A. Biot, Ann. Soc. Sci. **B54**, 91–109 (1934).
[3] M.A. Biot, Phil. Mag. Ser.7 **27**, 89–115 (1939).
[4] M.A. Biot, *Mechanics of incremental deformations*, John Wiley, New York (1965).
[5] J.F. Bell, Int. J. Plasticity **1**, 3–27 (1985).
[6] M.F. Beatty, In: *Nonlinear Elasticity: Theory and Applications*. Eds: Y.B. Fu and R.W. Ogden, 58–134 (Cambridge University Press, London 2001).
[7] M. Destrade, Int. J. Nonlinear Mech. (to appear 2002).
[8] V.G. Mozhaev, In: *IUTAM Symposium on anisotropy, inhomogeneity and nonlinearity in solids*. Eds: D.F. Parker and A.H. England, 455–462 (Kluwer, Holland 1995).
[9] M.A. Dowaikh and R.W. Ogden, IMA J. Appl. Math. **44**, 261–284 (1990).
[10] M. Romeo, J. Acoust. Soc. Am. **110**, 59–67 (2001).
[11] M.F. Beatty and F.X. Pan, Int. J. Non-Linear Mech. **33**, 867–906 (1998).
[12] M.F. Beatty and M.A. Hayes, J. Elasticity **29**, 1–84 (1992).
[13] M.F. Beatty and M.A. Hayes, Q. Jl. Mech. Appl. Math. **45**, 663–709 (1992).
[14] M.F. Beatty and M.A. Hayes, In: *Nonlinear Waves in Solids*. Eds: J.L. Wegner and F.R. Norwood, 67–72 (ASME 1995).
[15] M. Hayes and R.S. Rivlin, Arch. Rational Mech. Anal. **8**, 358–380 (1961).
[16] F.X. Pan and M.F. Beatty, Int. J. Non-Linear Mech. **34**, 169–177 (1999).
[17] Lord Rayleigh, Proc. R. Soc. London **A17**, 4–11 (1885).