EXPLICIT SECULAR EQUATIONS FOR SURFACE AND INTERFACE WAVES IN ANISOTROPIC SOLIDS

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Summary The derivation of secular equations in closed form for acoustic waves propagating at the interface of semi-infinite elastic bodies is made possible, using a simple method.

INTRODUCTION

Consider an inhomogeneous plane wave propagating in a semi-infinite anisotropic elastic solid with speed v and wave number k in the direction x_1 (say), and with attenuation in the direction x_2 (say), orthogonal to x_1. Its mechanical displacement is modeled as

$$\mathbf{u} = \mathbf{U}(kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \quad \mathbf{U}(\infty) = \mathbf{0}.$$

Anisotropic stress-strain relations, $\sigma_{ij} = c_{ijkl}u_{l,k}$ where $\mathbf{c}$ is a constant fourth-order elasticity tensor, imply that the stress components are of the form,

$$\sigma_{ij} = \mathrm{i}kt_{ij}(kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \quad t_{ij}(\infty) = 0.$$

The equations of motion can be written as a first-order differential system for the components of the displacement-traction vector $\boldsymbol{\xi}$,

$$\boldsymbol{\xi}' = i\mathbf{N}\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{\xi}(kx_2) := [U_1, U_2, U_3, t_{12}, t_{22}, t_{32}]^{\mathrm{T}}. \tag{1}$$

Here $\mathbf{N}$ is a real 6×6 matrix, whose components depend on the elastic constants and mass density characteristic of the material, and on the speed v. Finally, some boundary conditions are imposed at the interface $x_2 = 0$ for some (or all) components of $\boldsymbol{\xi}(0)$,

$$f(U_i(0), t_{i2}(0)) = 0, \tag{2}$$

such as for instance the vanishing of the tractions for a solid/vacuum interface (Rayleigh waves) or the continuity of the displacements and of the tractions for a solid/solid interface (Stoneley waves).

The usual method of resolution of (1)-(2) consists in the following steps. First take the solution to (1) in exponential form, $\boldsymbol{\xi}(kx_2) = \boldsymbol{\xi}^{\mathbf{0}}\mathrm{e}^{\mathrm{i}kpx_2}$. Then find the attenuation factors p_j as roots of: det $(\mathbf{N} - p\mathbf{I}) = 0$, $\Im(p) > 0$, and the partial waves $\boldsymbol{\xi}^{\boldsymbol{j}}$ as eigenvectors of: $\mathbf{N}\boldsymbol{\xi}^{\boldsymbol{j}} = p_j\boldsymbol{\xi}^{\boldsymbol{j}}$. Finally use the general solution $\boldsymbol{\xi}(kx_2) = \sum \gamma_j \boldsymbol{\xi}^{\boldsymbol{j}}\mathrm{e}^{\mathrm{i}kp_jx_2}$, to write (2): then a homogeneous system of equations with unknowns γ_j arises, and the corresponding determinantal equation is the secular equation, with v as the sole unknown. This approach was introduced by Stroh [1] and later used by Barnett & Lothe and others to address and answer many outstanding theoretical questions about the existence and uniqueness of a solution, bounds on the wave speed, etc. Moreover, Barnett & Lothe [2] also developped an "integral formalism" which yields efficient numerical schemes for the determination of the wave speed without having to compute the p_j. However this method is not appropriate in general to derive a secular equation explicitly. Indeed, only when the wave is polarized in the sagittal plane and certain elastic constants vanish can the p_j (and thus the secular equation) be found explicitly, as the roots of a biquadratic (Royer & Dieulesaint [3] identified the corresponding 16 configurations for solids with rhombic, tetragonal, cubic, and hexagonal symmetries.) Otherwise, the equation det $(\mathbf{N} - p\mathbf{I}) = 0$ is a bicubic, a quartic, or even a sextic for p, leading to an involved analysis in the first and second cases, or to an unsolvable problem in the latter case.

Hence a different procedure must be adopted. This search was initiated by Currie [4] and completed by Taziev [5] for Rayleigh waves, using some results of the Stroh formalism. Here a generalization is proposed for this and other types of interface waves (Stoneley waves, Scholte waves), without relying on the Stroh formalism.

FUNDAMENTAL EQUATIONS

The properties of the matrix $\mathbf{N}$ in (1) are well established. In particular, it can be checked that $\widehat{\mathbf{I}}\mathbf{N}^n$, where $\widehat{\mathbf{I}}$ is defined below and n is an integer, is a *symmetric* matrix with the following block structure,

$$\widehat{\mathbf{I}}\mathbf{N}^n = \begin{bmatrix} \mathbf{K}^{(\mathbf{n})} & \mathbf{N}_{\mathbf{1}}^{(\mathbf{n})\mathrm{T}} \\ \mathbf{N}_{\mathbf{1}}^{(\mathbf{n})} & \mathbf{N}_{\mathbf{2}}^{(\mathbf{n})} \end{bmatrix} = (\widehat{\mathbf{I}}\mathbf{N}^n)^{\mathrm{T}}, \quad \widehat{\mathbf{I}} := \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix}.$$

Here, $\mathbf{K}^{(\mathbf{n})}$ and $\mathbf{N}_{\mathbf{2}}^{(\mathbf{n})}$ are symmetric 3×3 matrices, and $\mathbf{I}$ is the 3×3 identity matrix. Thus, multiplying both sides of (1) by $\widehat{\mathbf{I}}\mathbf{N}^n\overline{\boldsymbol{\xi}}$ and adding the complex conjugate yields $\overline{\boldsymbol{\xi}} \cdot \widehat{\mathbf{I}}\mathbf{N}^n\boldsymbol{\xi}' + \overline{\boldsymbol{\xi}}' \cdot \widehat{\mathbf{I}}\mathbf{N}^n\boldsymbol{\xi} = 0$, and so by integration, $\overline{\boldsymbol{\xi}} \cdot \widehat{\mathbf{I}}\mathbf{N}^n\boldsymbol{\xi} =$ const. $= 0$, its value at infinity. In particular,

$$\overline{\boldsymbol{\xi}}(0) \cdot \widehat{\mathbf{I}}\mathbf{N}^n\boldsymbol{\xi}(0) = 0. \tag{3}$$

These *fundamental equations*, valid for any positive or negative values of the integer n, are sufficient to solve many problems of interface waves. Because of the Cayley-Hamilton theorem, if $\mathbf{N}$ is a 6×6 matrix then there are 5 independent fundamental equations, and if $\mathbf{N}$ is a 4×4 matrix (decoupling of in-plane from anti-plane strain and stress) then there are 3 independent equations. Examples solved so far are now briefly presented.

EXAMPLES

Rayleigh waves

For a solid/vacuum interface, the boundary conditions at $x_2 = 0$ are: $\boldsymbol{\xi}(0) = [\mathbf{U}(0), \mathbf{0}]^{\mathrm{T}}$, and (3) reduce to [4,5],

$$\overline{\mathbf{U}}(0) \cdot \mathbf{K}^{(\mathbf{n})}\mathbf{U}(0) = 0. \quad (4)$$

Then the secular equation is found explicitly for a completely arbitrary direction of propagation. Moreover, when the wave travels along a crystallographic axis of a rhombic crystal or along a principal direction of a pre-stressed hyperelastic material, then the body can be put into uniform rotation (gyroscopes, tires, ...) along one of the crystallographic/principal axes and the secular equation can also be found [6] (see Fig. 1(a)) ; in that case $\mathbf{K}^{(\mathbf{n})}$ is Hermitian and (4) still applies.

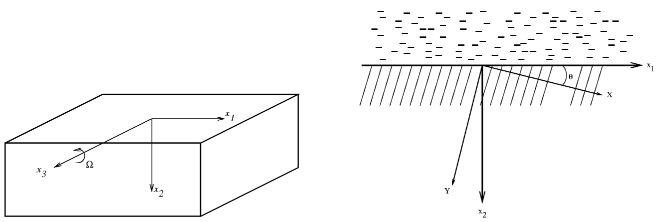

Figure 1. (a) Rayleigh wave in a rotating body; (b) Scholte wave polarized in a plane of symmetry.

Scholte waves

For a solid/fluid interface, the boundary conditions at $x_2 = 0$ are: $\boldsymbol{\xi}(0^+) = [U_1(0^+), U_2(0^+), U_3(0^+), 0, t_{22}(0^+), 0]^{\mathrm{T}}$ in the solid and: $t_{22}(0^-) = -\mathrm{i}ZU_2(0^-)$ in the fluid, where Z is the (real) normal impedance of the inviscid fluid. The continuity of U_2 and t_{22} accross the interface, combined with (3), yield the secular equation for waves either polarized in a symmetry plane [7] (see Fig. 1(b)) or propagating in a symmetry plane.

Stoneley waves

For a solid/solid interface, the boundary conditions at $x_2 = 0$ are: $\boldsymbol{\xi}(0^+) = \boldsymbol{\xi}(0^-)$. The fundamental equations (3) yield the secular equation when the semi-infinite bodies are made of same crystal [8] (see Fig. 2) , or of the same hyperelastic material subject to the same pre-stress [9], but with misaligned crystallographic/principal axes.

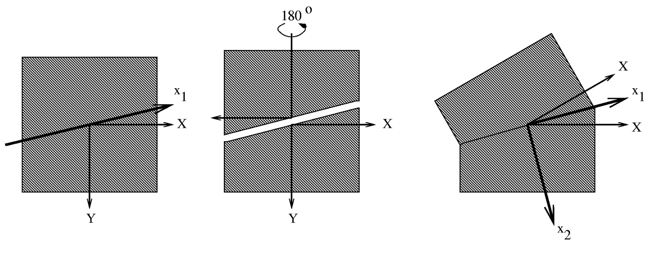

Figure 2. Cutting, rotating, and bonding of a rhombic crystal; a Stoneley wave exists at $x_2 = 0$.

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