A NOTE ABOUT WAVES IN DISSIPATIVE AND DISPERSIVE SOLIDS

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We study shear waves propagating in a special viscoelastic model proposed first by Fosdick and Yu in 1996. We deduce an asymptotic approximation which reduces the full balance equations to a system of evolution equations which are a vectorial generalization of the Modified KDV-Burger equation. In such a way we show that the model takes into account not only dissipative effects but also dispersive effects.

1. Introduction

We consider a transverse wave, polarized in the (XY) plane, and propagating in the Z direction, of a Cartesian coordinate system associated with an unbounded solid,

$$x = X + u(z,t), \qquad y = Y + v(z,t), \qquad z = Z. \tag{1}$$

Here u and v are the unknown scalar functions describing the motion. We compute the usual geometrical quantities of interest, here the left Cauchy-Green strain tensor and its inverse,

$$\mathbf{B} = \begin{bmatrix} 1+u_z^2 & u_z v_z & u_z \\ u_z v_z & 1+v_z^2 & v_z \\ u_z & v_z & 1 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -u_z \\ 0 & 1 & -v_z \\ -u_z & -v_z & 1+(u_z^2+v_z^2) \end{bmatrix}. \tag{2}$$

For a general hyperelastic, incompressible, isotropic material, we have the following *constitutive equation* for the Cauchy stress tensor $\mathbf{T}$ (see Chadwick[1] for instance),

$$\mathbf{T} = -p\mathbf{I} + 2(\partial\Sigma/\partial I_1)\mathbf{B} - 2(\partial\Sigma/\partial I_2)\mathbf{B}^{-1}, \tag{3}$$

where $\mathbf{I}$ is the identity tensor and $p = p(\mathbf{x}, t)$ is the arbitrary Lagrange multiplier associated with the constraint of incompressibility ($\sqrt{\det \mathbf{B}} = 1$ at all times). In Eq. (3), Σ is the strain energy function, which for an incompressible material depends only on I_1 and I_2, the first two principal invariants of $\mathbf{B}$: $I_1 \equiv \text{tr } \mathbf{B}$, $I_2 \equiv [I_1^2 - \text{tr } (\mathbf{B}^2)]/2$. For the motion of Eq. (1), the first two invariants are

$$I_1 = I_2 = 3 + u_z^2 + v_z^2. \tag{4}$$

The *equations of motion*, in the absence of body forces, are given in current form as div $\mathbf{T} = \rho\ddot{\mathbf{x}}$, where ρ is the mass density; here they read

$$-\frac{\partial p}{\partial x} + \frac{\partial T_{13}}{\partial z} = \rho u_{tt}, \qquad -\frac{\partial p}{\partial y} + \frac{\partial T_{23}}{\partial z} = \rho v_{tt}, \qquad \frac{\partial T_{33}}{\partial z} = 0. \tag{5}$$

Differentiating these equations with respect to x, we find $p_{xx} = p_{yx} = p_{zx} = 0$, so that $p_x = p_1(t)$, say. Similarly, by differentiating the equations with respect to y, we find $p_y = p_2(t)$, say. The first two equations in (5) reduce to

$$-p_1(t) + (Qu_z)_z = \rho u_{tt}, \qquad -p_2(t) + (Qv_z)_z = \rho v_{tt}, \tag{6}$$

and the third equation determines p. Here, $Q = Q(u_z^2 + v_z^2)$ is the *generalized shear modulus* of nonlinear elasticity, defined by

$$Q = 2(\partial\Sigma/\partial I_1 + \partial\Sigma/\partial I_2). \tag{7}$$

Following Destrade and Saccomandi[2], we take the derivative of Eqs. (6) with respect to z, we introduce the notations $U = u_z$, $V = v_z$, and the complex function $W = U + \mathrm{i}V$, and we recast Eq. (6) as the single complex equation

$$(QW)_{zz} = \rho W_{tt}, \tag{8}$$

where Q is now a function of $U^2 + V^2$ alone, $Q = Q(U^2 + V^2)$. In nonlinear elasticity, it is common to require that the acoustic tensor be positive, in which case (8) is an hyperbolic system.

Now we decompose the complex function W into modulus Ω and argument θ as

$$W(z, t) = \Omega(z, t)\exp(\mathrm{i}\theta(z, t)), \tag{9}$$

and we focus on *travelling wave solutions* in the form

$$W = W(s) = \Omega(s)\mathrm{e}^{\mathrm{i}\theta(s)}, \quad \text{where} \quad s = z - ct, \tag{10}$$

c being the speed. This ansatz reduces Eq. (8) to $(QW)'' = \rho c^2 W''$. We integrate it twice taking each integration constant to be zero in order to eliminate the rigid and the homogeneous motions. We end up with the simple equation,

$$\left(Q - \rho c^2\right) W = 0. \tag{11}$$

Eq.(11) says, in accordance with the theory of hyperbolic second order nonlinear equations, that waves of permanent form are impossible unless $Q - \rho c^2 = 0$. This last opportunity is possible only if Q is independent of z, a situation that may happen only (a) for special classes of constitutive equations or (b) for a special classes of initial data.

For the first possibility (a), we determine that the most general constitutive class for which the generalized modulus is independent of z is the *Mooney-Rivlin model*, for which $2\Sigma = C(I_1 - 3) + E(I_2 - 3)$, where C and E are positive material constants; then $Q = C + E = \text{const}$. This is a special case of a general result by Ruggeri[3] about the existence of a double exceptional wave in unconstrained isotropic elastic materials.

For the second possibility (b), Carroll[4] determined the special solutions known as *circularly polarized harmonic waves*,

$$u(z,t) = A\cos k(z - ct), \quad v(z,t) = \pm A \sin k(z - ct), \tag{12}$$

where A and k are arbitrary constants. For these motions, $U^2 + V^2 = A^2k^2$ and therefore Q is independent of z; then the equation of motion Eq. (11) leads to the following dispersion equation,

$$Q(A^2k^2) = \rho c^2, \tag{13}$$

which may be solved for any *reasonable* constitutive equation.

Smooth solutions of initial-value problems for nonlinear hyperbolic systems are rare. Usually singularities will develop after a finite time, even when the initial data are smooth. To the best of our knowledge, theorems of global-in-time well-posedness to the initial-value problem for quasi-linear wave equations may be achieved only under the assumption of small initial data and the additional *null condition*[5]. Since our solutions (12) are smooth also for arbitrary large initial-data, it is clear that our knowledge of the mathematics of hyperbolic systems is still incomplete.

Kolsky[6] and Mason[7] showed that it is possible to produce experimentally tensile waves in stretched natural rubber bands, where the front becomes sharper as they progress. This is because natural vulcanized rubber becomes increasingly stiff with increasing tensile stress and because the stress-strain curve changes from concave to convex when very large deformations are involved. We argue that the hyperbolic system (8) may be a good mathematical approximation of such shock-like phenomena (For a more recent account of similar experiments, we refer to Vermorel et al.[8].)

On the other hand, rubber-like materials exhibit strong attenuation in the usual range of applications; it is for this reason that rubber is often used to damp out vibrations and to absorb shocks. Moreover, the underlying microstructure of polymeric materials introduces a characteristic nonlocal scale which is nearly 30 times the characteristic scale of face centered cubic materials such as copper[9]. This means that in many interesting applications there is an important range of wave-lengths where wave phenomena in elastomers and soft-tissues must be dispersive.

Thus it is necessary to improve our mathematical models of dynamic phenomena in rubber-like materials and to take into account both *dissipation* and *dispersion*. The aim of the present Note is to introduce a simple model accounting for these two effects in the framework of nonlinear solid mechanics.

2. Dispersion and dissipation

Guided by preliminary work[2], we now augment the constitutive equation Eq. (3) to $\mathbf{T} + \mathbf{T}^{\mathrm{D}}$, with

$$\mathbf{T}^{\mathrm{D}} = \nu \mathbf{A}_1 + \alpha[\mathbf{A}_2 - \mathbf{A}_1^2], \tag{14}$$

where $\mathbf{D}$ is the stretching tensor, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors,

$$\mathbf{D} \equiv (\mathbf{L} + \mathbf{L}^{\mathrm{T}})/2, \qquad \mathbf{A}_1 \equiv 2\mathbf{D}, \qquad \mathbf{A}_2 \equiv \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_1. \tag{15}$$

The *viscosity function* $\nu = \nu(\mathbf{D} \cdot \mathbf{D})$, and the *dispersion material function* $\alpha = \alpha(\mathbf{D} \cdot \mathbf{D})$, must be positive due to thermodynamics restrictions.

Destrade and Saccomandi[10], show that at $\nu \equiv 0$, Eq. (14) coincides exactly with the dispersion function proposed by Rubin et al.[11], and that $\mathbf{T}^{\mathrm{D}}$ is a straightforward generalization of the extra Cauchy stress tensor associated with a non-Newtonian fluid of second grade[12], which is

$$\nu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{16}$$

where ν is the classical viscosity and α_1, α_2 are the microstructural coefficients.

In the case of the motion (1), the kinematical quantities of interest are $\mathbf{A}_1$, $\mathbf{A}_1^2$, and $\mathbf{A}_2$, given by

$$\begin{bmatrix} 0 & 0 & u_{zt} \\ 0 & 0 & v_{zt} \\ u_{zt} & v_{zt} & 0 \end{bmatrix}, \qquad \begin{bmatrix} u_{zt}^2 & u_{zt}v_{zt} & 0 \\ u_{zt}v_{zt} & v_{zt}^2 & 0 \\ 0 & 0 & u_{zt}^2 + v_{zt}^2 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & u_{ztt} \\ 0 & 0 & v_{ztt} \\ u_{ztt} & v_{ztt} & 2(u_{zt}^2 + v_{zt}^2) \end{bmatrix}, \tag{17}$$

respectively. Hence we find that

$$\mathbf{D} \cdot \mathbf{D} = \tfrac{1}{2}(u_{zt}^2 + v_{zt}^2). \tag{18}$$

Following the process conducted in Section 2 for hyperelastic materials, and introducing the complex function $W \equiv U + \mathrm{i}V$, we recast the determining equations for the transverse wave motions as the following single complex equation,

$$(QW)_{zz} + (\nu W_t + \alpha W_{tt})_{zz} = \rho W_{tt}, \tag{19}$$

where Q is again a function of $U^2 + V^2 = \Omega^2$ alone: $Q = Q(\Omega^2)$, and ν and α are now functions of $U_t^2 + V_t^2$ alone: $\nu = \nu(U_t^2 + V_t^2)$, $\alpha = \alpha(U_t^2 + V_t^2)$.

3. A vector evolution equation

As is usual, we now perform a *moving frame expansion* for equation Eq. (19), with the new scales $s = z - ct$, $\tau = \epsilon t$ (here ϵ is a small parameter).

We assume that W is of the form

$$W = \epsilon^{1/2} w, \quad \text{where} \quad w = O(1). \tag{20}$$

Then $\Omega = |W| = \epsilon^{1/2}|w|$ and we expand the terms in (8) as

$$\begin{aligned} (QW)_{zz} &= \epsilon^{1/2} Q(0) w_{ss} + \epsilon^{3/2} Q'(0) \left(|w|^2 w\right)_{ss} + \dots, \\ \rho W_{tt} &= \epsilon^{1/2} \rho c^2 w_{ss} - 2\epsilon^{3/2} \rho c w_{s\tau} + \dots, \\ (\nu W_t)_{zz} &= -\epsilon^{1/2} c \nu(0) w_{sss} + \epsilon^{3/2} \nu(0) w_{ss\tau} + \dots, \\ (\alpha W_{tt})_{zz} &= \epsilon^{1/2} c^2 \alpha(0) w_{ssss} - 2\epsilon^{3/2} c \alpha(0) w_{sss\tau} + \dots \end{aligned} \tag{21}$$

In order to recover the linear wave speed at the lowest order (here, $\epsilon^{1/2}$) given by $Q(0) = \rho c^2$, we must assume that

$$Q(0) = O(1), \quad \nu(0) = O(1), \quad \text{and} \quad \alpha(0) = O(\epsilon) = \epsilon \alpha_0 \text{ (say)}, \tag{22}$$

where α_0 is a constant of order $O(1)$.

Then we find that at the next order,

$$Q'(0)\left(|w|^2 w\right)_{ss} - c\nu(0)w_{sss} + c^2\alpha_0 w_{ssss} = -2\rho c w_{s\tau}, \tag{23}$$

which we integrate once with respect to s to get the *vectorial MKdV-Burgers equation*,

$$w_\tau + q\left(|w|^2 w\right)_s - n w_{ss} + p w_{sss} = 0, \tag{24}$$

where $q \equiv Q'(0)/(2\rho c)$, $n \equiv \nu(0)/(2\rho)$, and $p \equiv c\alpha_0/(2\rho)$.

In this equation, the third derivative term is associated with dispersive phenomena whereas the second derivative term is associated with dissipative phenomena. To derive (24), we assumed that the nonlinear elastic effects are of the same order as the dissipative effects whilst the dispersive effects are of smaller order than elastic and dissipative effects. This assumption is quite realistic in biological applications at the length scales of interest in the framework of elastography.

4. Travelling waves solutions

We search for *travelling wave solutions* to equation (24). Introducing the variable $\xi = s - v\tau$, where v is the speed in the moving frame, we reduce (24) to the ordinary differential equation

$$-vw' + q\left(|w|^2 w\right)' - nw'' + pw''' = 0, \tag{25}$$

where a prime denotes the derivative with respect to ξ. With the usual asymptotic boundary conditions, we integrate once to obtain

$$\left(q|w|^2 - v\right) w - nw' + pw'' = d, \tag{26}$$

(here d is a real integration constant, to be considered null if we are interested in drop boundary conditions). Then, separating the real part of this equation from the imaginary part, by using the notation $w(\xi) = \omega(\xi)\mathrm{e}^{\mathrm{i}\vartheta(\xi)}$ say, gives

$$p\left(\omega'' - \omega\vartheta'^2\right) - n\omega' + \left(q\omega^2 - v\right)\omega = d, \qquad \omega^2\vartheta' = I\mathrm{e}^{\frac{n}{p}\xi}, \tag{27}$$

where the imaginary part has been integrated directly, with I as the integration constant. If we set $n = 0$ (no dissipation) in (27), then we recover a classical result by Gorbatcheva and Ostrosky[13].

The system (27) may be reduced to a single non-autonomous second order equation in the general case ($I \neq 0$.) When we consider linearly-polarized waves ($I = 0$) we get an autonomous Duffing-like equation with damping

$$\omega'' - \frac{n}{p}\omega' + \left(\frac{q}{p}\omega^2 - \frac{v}{p}\right)\omega = d. \tag{28}$$

In fact, if we focus on linearly polarized waves from the outset, then $|w| = w$, and Eq. (24) is the scalar MKdV-Burgers equation,

$$w_\tau + \left(w^3\right)_x = \alpha w_{xx} + \beta w_{xxx}, \tag{29}$$

where $x \equiv s/q$, $\alpha = n/q^2$, and $\beta = -p/q^3$. The possibility of travelling wave solutions to the modified Korteweg-deVries-Burgers equation has been studied in great details over the years, in particular by Jacobs et al.[14], Wang[15], Feng[16], or Vladimirov et al.[17]. However it seems that the combination: $\alpha > 0$, $\beta < 0$, — as found here — precludes the possibility of exact solutions.

We recall that the governing equations derived above have been obtained for any type of nonlinear incompressible solid, for which dispersion and dissipation can be modelled by Eq. (14); however, the equations were the result of a small parameter expansion, see Section 3. We now remark that it is also possible to study travelling waves for the *exact equation* (19), both in the case of linearly-polarized waves and in the non-linearly polarized case, when a given constitutive behaviour is chosen. For instance, we make the constitutive assumptions that $\alpha =$ const., $\nu =$ const., which are the simplest assumptions we can make for the modelling of dispersion and dissipation. For the shear modulus, the choice $Q =$ const. corresponds to a Mooney-Rivlin type of elastic behaviour and it has been treated elsewhere[18]: it leads to linear differential equations. Then the assumption $Q = \mu_0 + \mu_1\Omega^2$ (where μ_0, μ_1 are positive constants) is the simplest one we can make to uncover nonlinear governing equations; it corresponds to fourth-order elasticity theory[10]. With these assumptions, equation (19) reduces to

$$\left[(\mu_0 + \mu_1\Omega^2)W\right]_{zz} + \nu W_{tzz} + \alpha W_{ttzz} = \rho W_{tt}, \tag{30}$$

a vectorial version of the damped good Boussinesq equation. A thorough review on several mechanical aspects and applications of this equation may be found in the recent paper by Christov et al.[19]. When we study travelling waves of (30) in the linearly polarized case, we obtain

$$\alpha c^2\Omega'' - \nu c\Omega' + \left(\mu_1\Omega^2 + \mu_0 - \rho c^2\right)\Omega = D, \tag{31}$$

which is equivalent to (28).

By standard phase plane analysis it is possible to obtain the conditions on the coefficients of (31) for which *kink*-like solutions are possible. For example in the case $D = 0$ (or $d = 0$ if we are considering (28)) a detailed discussion of these solutions is provided by Feng[20] where it is shown that monotonous kinks (like in dissipative systems) are possible when the classical viscosity is sufficiently strong. Otherwise, when dispersive and/or nonlinear elastic effects are more important than dissipative effects, we observe kink-like solutions with an oscillatory character.

References

1. P. Chadwick, *Continuum Mechanics* (Dover, New York, 1999).
2. M. Destrade and G. Saccomandi, *Phys. Rev. E* **72**, 016620 (2005).
3. T. Ruggeri,, *Annali di Matematica pura ed applicata (IV)* **CXII**, 315 (1977).
4. M. M. Carroll, *Q. J. Mech. appl. Math.* **30**, 223 (1977).
5. W. Domański and R.W. Ogden *Arch. Mech* **58**, 339 (2006).
6. H. Kolsky, *Proc. Phys. Soc. B* **62**, 676 (1949).
7. P. Mason, *Proc. R. Soc. A* **272**, 315 (1963).
8. R. Vermorel, N Vandeberghe, and E. Villermaux *Proc. Roy. Soc. A* **463**, 641-658 (2007).
9. R. Marangani and P. Sharma, *Phys. Rev. Lett.* **98**, 195504 (2007).
10. M. Destrade and G. Saccomandi, *Wave Motion* (to appear 2007).
11. M. B. Rubin, P. Rosenau and O. Gottlieb, *J. Appl. Phys.*, **77**, 4054 (1995).
12. R. L. Fosdick and J. H. Yu, *Int. J. Nonlinear Mech.* **31**, 495 (1996).
13. O.B. Gorbacheva, L.A. Ostrovsky, *Physica D*, **8**, 223 (1983).
14. D. Jacobs, B. McKineey, and M. Shearer, *J. Diff. Eq.* **116**, 448 (1995).
15. M. Wang, *Phys. Lett. A* **213**, 279 (1996).
16. Z. Feng, *Phys. Lett. A* **318** 522 (2003).
17. V.A. Vladimirov, E.V. Kutafina, and A. Pudelko, *SIGMA* **2**, Paper 061 (2006).
18. M. Destrade and G. Saccomandi, *Proc. 13th Conf. Waves Stab. Continuous Media WASCOM 2005* (World Scientific, 2006).
19. C.C. Christov, G.A. Maugin, and A.V. Porubov, *C. R. Mécanique* doi:10.1016/j.crme.2007.08.006 (2007).
20. Z. Feng, *Chaos, Solitons and Fractals* **28** 463 (2006).