Euler-type Buckling Instabilities in the Pure Bending of a Thick Rubber Block

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ABSTRACT

Our main aim in the present investigation is to revisit the pure bending of a thick rubber block deforming in plane strain, a problem that was first investigated in [1]. In particular, we show that for an incompressible neo-Hookean material the instability is of Euler type, with a well defined characteristic wavelength. The surface instabilities reported in [1] for different constitutive assumptions are likely to be erroneous, for reasons that we shall explain.

The undeformed geometry is assumed to be characterised by three parameters: $2L$ (*length*), H (*height*) and $2A$ (*thickness*). Due to the plane strain hypothesis, it is enough to confine our attention to the cross sections (shown shaded below) perpendicular to the vertical axis of the block and situated sufficiently far away from the lower and upper faces.

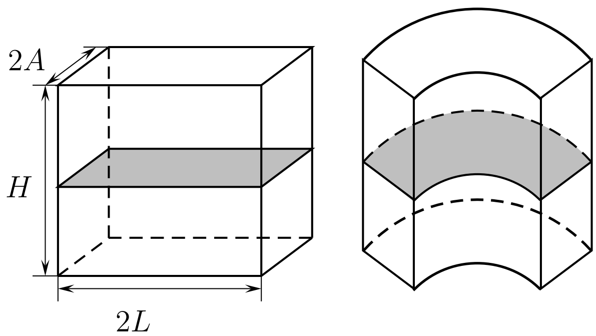

Figure 1: Cylindrical bending of a rubber block

The reference configuration $\{(X_1, X_2) \in \mathbb{R}^2 \mid -A \le X_1 \le A,\ -L \le X_2 \le L\}$ is mapped onto $\{(r,\theta) \in \mathbb{R} \times (-\pi, \pi] \mid r_1 \le r \le r_2,\ -\omega_0 \le \theta \le \omega_0\}$ via the pure bending deformation. It is well known that its expression is given by $r = (d + 2X_1/\omega)^{1/2}$, $\theta = \omega X_2$, where d is a quantity determined by the particular constitutive equation adopted and ω serves as a control parameter, being related to the angle of bending $\omega_0 \equiv \omega L \in [0, \pi)$. We use this deformation as the basic state on which we superimpose an infinitesimal perturbation. The stability equations that follow by linearising the two-dimensional field equations are recorded in [2], where it is showed further that they can be reduced to a single fourth-order PDE with variable coefficients

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by introducing an incremental displacement potential, $\phi(r,\theta)$. The analysis is simplified by looking for a normal mode solution $\phi(r,\theta) = \Phi(r)\cos(m\theta)$, where $m \in \mathbb{N}$ is the azimuthal mode number related to the number of ripples on the compressed (inner) face of the block; satisfying boundary conditions turns out to impose a constraint that takes the form $m = n\pi/\omega_0$, for some $n \in \mathbb{N}$.

Next, the bulk material is modelled by a simple neo-Hookean strain-energy function, $\mathcal{W}(\lambda_r,\lambda_\theta) \propto (\lambda_r^2 + \lambda_\theta^2 - 2)$, where the principal stretches λ_r and λ_θ are associated with the Eulerian principal directions. Due to the incompressibility constraint, these can be written as $\lambda_r \equiv \lambda^{-1}$ and $\lambda_\theta \equiv \lambda = \omega r$, which defines the notation λ. For this particular constitutive choice, d mentioned above is determined by $d = (L/\omega_0)^2(1 + 4\eta^2\omega_0^2)^{1/2}$, with $\eta \equiv H/L$.

After lengthy algebraic manipulations, the ODE satisfied by Φ can be showed to be

$$\rho\Phi'''' - 2\Phi''' + \mathcal{P}(\rho;\omega_0,n)\Phi'' + \mathcal{R}(\rho;\omega_0,n)\Phi' + \rho n^4\pi^4\Phi = 0, \quad \text{in} \quad \rho_1(\omega_0,\eta) < \rho < \rho_2(\omega_0,\eta)\,,$$

where $\rho = r/L$, $\mathcal{P}(\rho;\omega_0,n) = \mathcal{O}(n^2) = \mathcal{R}(\rho;\omega_0,n)$ as $n \to \infty$, and $\rho = \rho_j$ $(j = 1,2)$ represent the equations of the two curved boundaries of the bent rubber block; we note also that the principal stretch in the $\mathbf{e}_\theta$-direction assumes the simple form $\lambda = \omega_0\rho$. In solving the eigenproblem for the pure bending eigenmodes not only do we have to find λ, but the parameter n must also be identified such that $\lambda_1 \equiv \lambda(\rho_1)$ is maximum. Our direct numerical simulations (details of which can be found in [2]) showed that the number of ripples on the compressed side of the block increases with the non-dimensional thickness η. When this latter quantity is reasonably large ($\eta \gtrsim 3$) the critical mode number m always corresponds to $n = 1$. The conclusion that emerges is that the behaviour of a very thick block can be understood in two different ways: either (i) assuming that $n = 1$ and $\eta \gg 1$ or (ii) fixing $\eta = \mathcal{O}(1)$ and letting $n \gg 1$. We have pursued the latter alternative and showed that asymptotic results can be obtained easily by using classical boundary-layer theory or WKB methods; such approaches lead to the following expressions for the *critical bending angle* (ω_0) or the *critical stretch* (λ_1)

$$\omega_0 = \Omega_0 + n^{-1}\Omega_1 + n^{-2}\Omega_2 + \dots\,,$$

$$\Omega_0 = 0.7718\,, \qquad \Omega_1 = -0.4156\,, \qquad \Omega_2 = 1.5600\,,$$

and

$$\lambda_1 = \Lambda_0 + n^{-1}\Lambda_1 + n^{-2}\Lambda_2 + \dots\,, \tag{1}$$

$$\Lambda_0 = 0.543689\,, \qquad \Lambda_1 = 0.1228\,, \qquad \Lambda_2 = -0.4240\,.$$

As expected, $\Lambda_0 \simeq 0.544$ represents the critical value of the principal stretch for the surface instability of a compressed neo-Hookean half-space (as originally found by M.A. Biot), while the next-order corrections in formula (1) account for the finite size of the rubber block.

References

[1] N. Triantafyllidis, Bifurcation phenomena in pure bending. *J. Mech. Phys. Solids* **28**, 221–245, 1980.

[2] C.D. Coman and M. Destrade, Asymptotic results for bifurcations in pure bending of rubber blocks. *Quart. Jl. Appl. Math. Mech.* **61**, 395–414, 2008.