Small-amplitude elastic waves in soft matter

Michel Destrade

Stokes Centre for Applied Mathematics,
School of Mathematics, Statistics and Applied Mathematics,
NUI Galway, University Road, Galway, Ireland.

Introduction

Biological soft tissues and soft gels are difficult to study and model mathematically. Bioengineers often see them as engineering materials and try to evaluate their mechanical properties with standard testing protocols, such as tensile testing, simple shear, torsion, etc. These processes are destructive for tissues, as a sample is taken out of the body and placed in a device. The resulting measured parameters and models are expected to be very different from their in vivo counterparts.

To test soft tissues properly, non-destructively, and non-invasively, we can rely on elastic waves. We can study the influence of pre-stress on their speed and obtain the nonlinear elastic parameters by inverse analysis. This idea forms the basis of the theory of *acousto-elasticity*, which can be dated back to early works of Brillouin, and has been used successfully in the past for "hard" elastic solids such as rocks and metals.

Here we explore the extension of acousto-elasticity to "soft" elastic solids, which can be subjected to large deformations in service. We look at theoretical, numerical, experimental, and even clinical results, generated in particular on gels, brain, breast, and skin.

Destructive testing of soft matter

To test soft biological tissue, we can rely on existing testing protocols for soft solids such as rubber. Hence we can, in principle, stretch, twist, shear, inflate, and bend them, but in practice it is not always possible to obtain reliable and accurate measurements. Soft tissues *ex vivo* have very different properties from when they are "in service", because of cell death, dehydration, release of residual stresses and many other factors. Moreover, they do not all lend themselves to be tested in those protocols.

For instance, brain matter is extremely soft and fragile, and cannot be tested as a thin membrane, which rules out the inflation, pure shear, and dog-bone tensile tests. It cannot be gripped or attached by hooks, and must be glued instead. That limitation rules out cylinder tension and compression tests, as inhomogeneous local effects then develop near the attached platens and makes modelling difficult. But within this limitation, *simple shear* and *torsion* do work reasonably well, see Figures 1(a)-(d).

Experimentally, we find linear relationships between the Cauchy shear stress component T_{12} and the amount of shear K, and between the torque τ and the twist ϕ, when we test pig brain samples in simple shear and torsion, see Figures 1(e)-(f). Using the general expressions

$$T_{12} = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)K, \qquad \tau = 4\pi\phi\int_0^a r^3\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)dr, \tag{1}$$

(where r is the radial distance and a is the radius of the twisted cylinder), we see that the Mooney-Rivlin strain energy density W_{MR} is particularly appropriate,

$$W_{\mathrm{MR}} = \tfrac{1}{2}C_1(I_1 - 3) + \tfrac{1}{2}C_2(I_2 - 3), \tag{2}$$

(where $C_1 > 0$, $C_2 > 0$ are constants, and $I_1 = \operatorname{tr}\mathbf{C}$, $I_2 = \operatorname{tr}(\mathbf{C}^{-1})$ are the first two principal invariants of the right Cauchy-Green deformation tensor $\mathbf{C}$), because it provides indeed exact linear relationships.

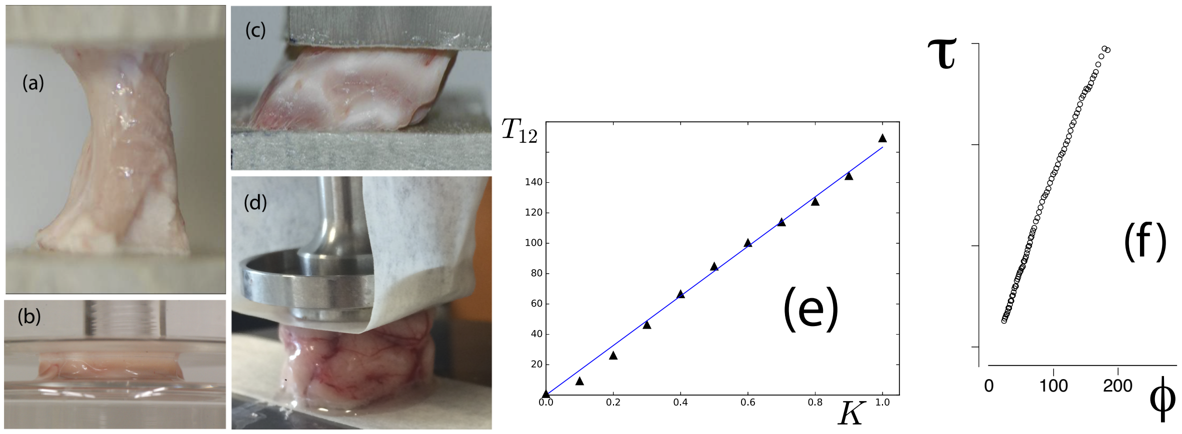

Figure 1: Standard testing protocols for soft solids applied to brain matter (porcine): (a) tensile test with glued ends and (b) compression test with lubricated faces: notice the inhomogeneity of the resulting deformations; (c) simple shear and (d) torsion: here the samples behave as required for the modelling. (e): shear stress Vs amount of shear [1]; (f): torque Vs twist [2].

Non-destructive testing with acousto-elasticity

To test soft tissues *in situ*, we may rely on *acousto-elasticity theory*, which gives the relationship between the speed of an elastic wave and the deformation of the soft solid.

For example, a homogeneous plane wave of small amplitude travels in an incompressible solid of mass density ρ and strain energy density W with speed v given by the formula

$$\rho v^2 = \left(\lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_1 \frac{\partial W}{\partial \lambda_1}\right) \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2}, \tag{3}$$

where the λs are the principal stretches along the Eulerian principal directions x_1, x_2, x_3 (specifically, the body wave travels in the x_2-direction and is polarised in the x_1- direction).

Hence consider measuring the speed of a shear wave in an intact brain, subject to a small-but-finite compression described by $\lambda_1 = 1+e$, $\lambda_2 = \lambda_3 = 1 - e/2 + 3e^2/8$, where e (small) is the amount of contraction (a contraction of 10% corresponds to $\lambda_1 = 0.9$, $e = -0.1$). Then the formula above gives [3],

$$\rho v^2 = \mu + (A/4)e + (2\mu + A + 3D)e^2, \tag{4}$$

where μ is the initial shear modulus, and A, D are the Landau coefficients of third- and fourth-order weakly non-linear elasticity, respectively. It is then a simple matter to estimate these material parameters, by collecting the $\rho v^2 - e$ data and fitting it to quadratic, see Figure 2.

With this talk we will explore further extensions of the theory of acousto-elasticity. First, with the propagation of waves in *thin-walled*, pre-stretched soft solids, which are commonly found in the

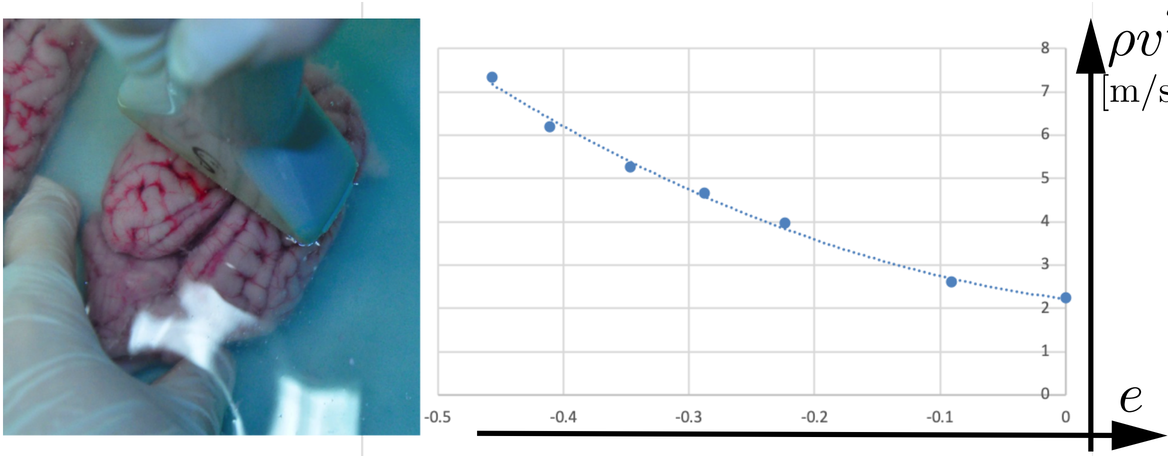

Figure 2: Using acousto-elasticity to find the material parameter of an intact pig brain. The curve fitting exercise on the right yields $\mu = 2.2$, $A = -15.0$, $D = 8.7$ kPa.

body (Achille's tendon, arterial walls, bladder, mitral valve, dura matter, etc.). In that case the mathematics becomes more involved because of dispersion but good and practical approximations can be found. Then, with waves travelling on the surface of the human *skin*, which, it turns out, must be modelled as a pre-strained, hyperelastic solid with two orthogonal embedded families of parallel fibres [4].

References

[1] Destrade, M., Gilchrist, M.D., Murphy, J.G., Rashid, B. and Saccomandi, G., 2015. Extreme softness of brain matter in simple shear. International Journal of Non-Linear Mechanics, 75, pp.54-58.

[2] Balbi, V., Trotta, A., Destrade, M., and Ní Annaidh, A. Poynting effect of brain matter in torsion, to appear.

[3] Jiang, Y., Li, G., Qian, L.X., Liang, S., Destrade, M. and Cao, Y., 2015. Measuring the linear and nonlinear elastic properties of brain tissue with shear waves and inverse analysis. *Biomechanics and Modeling in Mechanobiology*, 14(5), pp.1119-1128.

[4] Deroy, C., Destrade, M., Mc Alinden, A. and Ní Annaidh, A., 2017. Non-invasive evaluation of skin tension lines with elastic waves. Skin Research and Technology, 23(3), pp.326-335.