Groups I **MA343/MA532** **Problems II**

1. Let $C_{12} = < a : a^{12} = 1 >$ be a cyclic group of order 12. (Note: $a^k \neq 1$ for $0 < k < 12$.) Determine explicitly the (elements in the) cyclic subgroups $<a^4>$, $<a^5>$, $<a^8>$, $<a^{10}>$. Express each of the last three subgroups in the form $< a^m >$ for some **divisor** m of 12. What are the orders of the elements a^4, a^5, a^8 and a^{10} ?

2. Let $C_n = < a : a^n = 1 >$ be a cyclic group of (finite) order n. Show that if H is a subgroup of C_n then $H = < a^m >$ for some $m \mid n$. (So a subgroup of a finite cyclic group is cyclic!)

3. Let G be a *finite* multiplicative group. Show that a non-empty subset H of G is a subgroup if H is closed under the multiplication. Deduce that the set Alt(n) of all *even* permutations is a subgroup of the group Sym(n) of all permutations of $\{1, 2, 3, ..., n\}$.

4. Find the distinct cosets of H in G, and the index $|G : H|$, in each of the cases:
   (i) G is the group in Q3 of Problems I, and $H = < d >$;
   (ii) G is Alt(4) and $H = < (413) >$.

5. Prove the Theorem of Lagrange: if H is a subgroup of the finite group G then (its order) $|H|$ divides $|G|$.

6. Let G be a group of (finite) order n. Show that the order $o(a)$ of $a \in G$ is a divisor of n, and deduce that $a^n = 1$. Show also that (i) $o(a^{-1}) = o(a)$ and (ii) $a^{-1} = a \Leftrightarrow a = 1$ or $o(a) = 2$.

7. Show that if $a, b \in G$ satisfy $o(a) = o(b) = o(ab) = 2$ then $ba = ab$, and deduce that $H = \{1, a, b, ab\}$ is a (noncyclic) subgroup of order 4 in G. [Take care to check that the listed elements of H are *distinct*.]

8.(i) Assuming associativity, show that $G = \{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : p, q, r, s \in Z;\ ps - qr = 1 \}$ is a group under matrix multiplication.
   (ii) Find the elements of the subgroups $< a >$, $< b >$ and $< c >$ of G, where

$$a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Homework: 2, 4, 8. Due: 25 October.