GROUPS I MA343 Problems III

1. Show that a *non-cyclic* group G of order 6 contains an element, say x, of order 3 and an element, say y, of order 2. Deduce that $G = \{1, x, x^2, y, xy, x^2y\}$ and that $yx = x^2y$. Express yx^2 and $(xy)^2$ as elements in the list for G.

2. Consider the matrices $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Calculate $c = ab$ and check that $a^2 = b^2 = c^2 = -I$ and $ba = -c$. Deduce that $G = \{I, -I, a, -a, b, -b, c, -c\}$ is a group under matrix multiplication. Find the order of each element of G.

3. Let K be a normal subgroup of G. Define *normal* and show that the set G/K of cosets of K in G is a group under (set) multiplication. Show that $K = \{\pm I\}$ is normal in the group G in Q2 above, and calculate the multiplication table for G/K.

4. (i) Let G be a finite multiplicative group. Show that the *centre* Z of G, given by $Z = \{x \in G : \ xg = gx \text{ for all } g \in G\}$, is a *normal subgroup* of G.
   (ii) Consider the matrices $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Calculate $c = ab$ and check that $a^2 = -I$, $b^2 = c^2 = I$ and $ba = -c$. Deduce that $G = \{I, -I, a, -a, b, -b, c, -c\}$ is a group under matrix multiplication. Find the centre Z of G and the multiplication table for G/Z.

5. The map $f : G \to H$ is an isomorphism. Define *isomorphism* and show that $o(f(a)) = o(a)$ for all $a \in G$. Deduce that the matrix groups in Q2 and Q4 above are *not* isomorphic. Find (and verify) an isomorphism f between $Z_4, +$ and $\{\pm 1, \pm i\}, \times$ such that $f(3) = i$.

6. Let $f : G \to H$ be a homomorphism of finite multiplicative groups. Define *homomorphism* and show that: (i) $f(1_G) = 1_H$; (ii) $f(x)^{-1} = f(x^{-1})$ for all x in G; (iii) $f(G) = \{f(x) : x \in G\}$ is a subgroup of H; (iv) $K = \{x \in G : f(x) = 1_H\}$ is a *normal subgroup* of G; (v) the map $\bar{f} : G/K \to H$ given by $\bar{f}(Kx) = f(x)$ defines an *isomorphism* between G/K and $f(G)$.

7. Let a be an element of the finite multiplicative group G. Define the centraliser $C_G(a)$ and the conjugacy class $ccl_G(a)$. Show that $C_G(a) \le G$ and $|G| = |C_G(a)| \cdot |ccl_G(a)|$.

8. Describe briefly how the conjugacy classes in the symmetric group S_n correspond to partitions of n. Determine, with explanation, the *size* of the class $ccl_G(\alpha)$ and the *elements* of the centraliser $C_G(\alpha)$ when $G = S_5$ and (i) $\alpha = (415)$ (ii) $\alpha = (5432)$.