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Linearising Nonsmooth Flows with Stochastic Discontinuity Boundaries

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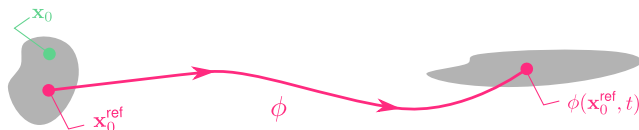
Linearisation



Suppose the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = \mathbf{x}_0, \quad \text{has the unique solution} \quad \mathbf{x}(t) = \phi(\mathbf{x}_0, t). \quad (1)$$

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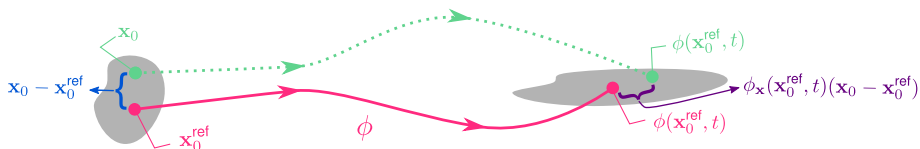


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$$\phi(\mathbf{x}_0, t) - \phi(\mathbf{x}_0^{\text{ref}}, t) = \phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)(\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}) + \mathcal{O}(\|\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}\|), \quad (2)$$

where the Jacobian $\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)$ is the solution to the IVP

$$\dot{\Phi} = \mathbf{f}_{\mathbf{x}}(\phi(\mathbf{x}_0^{\text{ref}}, t))\Phi, \quad \Phi(0) = \mathbf{I}. \quad (3)$$

Linearisation

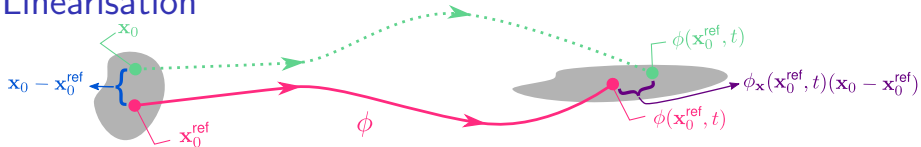


Figure: Linearisation of smooth dynamical systems

Linearisation

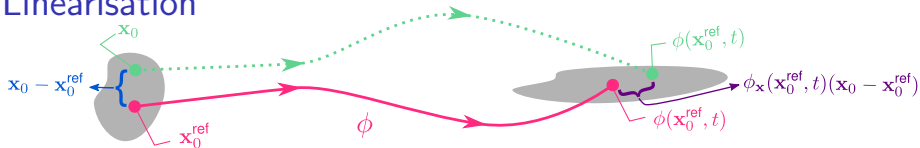


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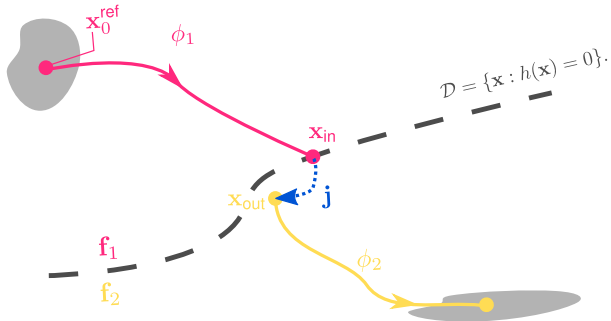


Figure: A nonsmooth dynamical system

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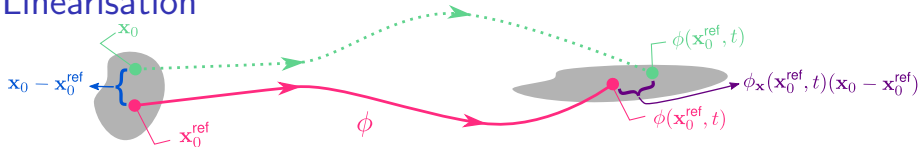


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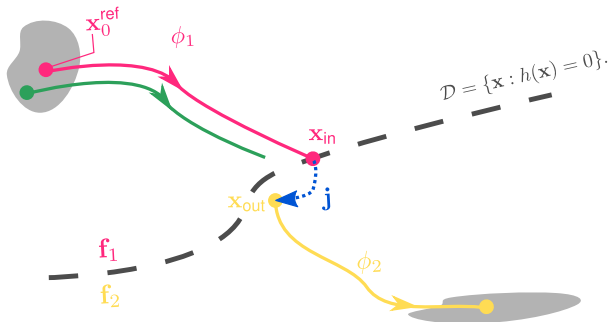


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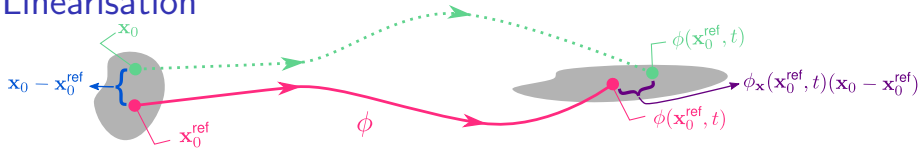


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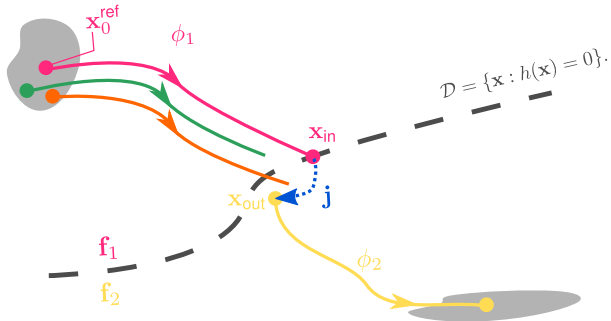


Figure: A nonsmooth dynamical system

Constructing the Zero-Time Discontinuity Mapping

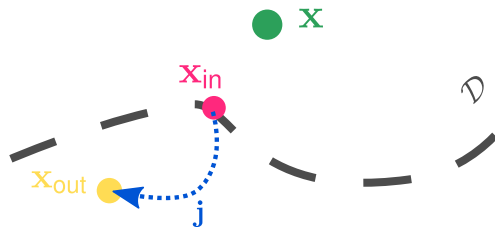


Figure: Constructing the ZDM

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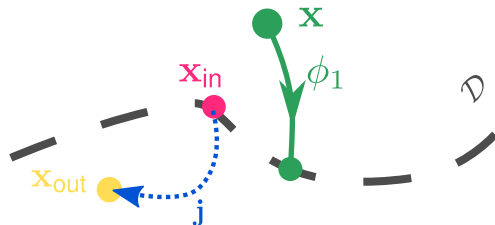


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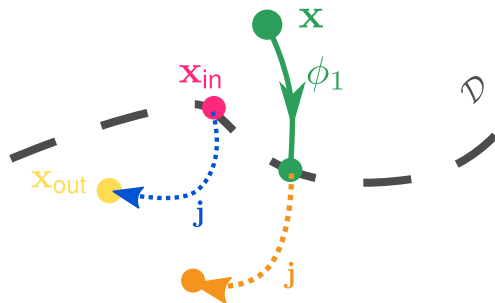


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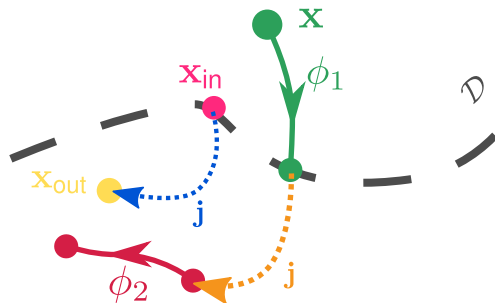


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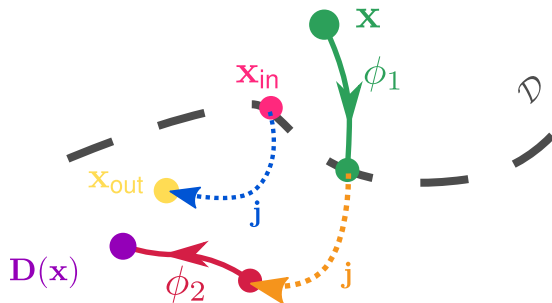


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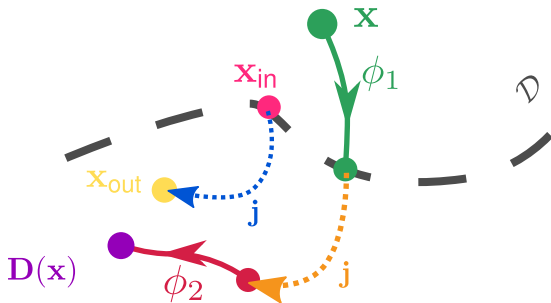


Figure: Constructing the ZDM

We can now write

$$\phi(\mathbf{x}_0, T) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), T - t_{\text{ref}}), \quad (4)$$

where the ZDM

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x})) \quad (5)$$

takes a point in a neighbourhood of \mathbf{x}_{in} and maps it to a point in a neighbourhood of \mathbf{x}_{out} .

The Saltation Matrix

The Jacobian of \mathbf{D} evaluated at \mathbf{x}_{in} is given by

$$\begin{aligned}\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{in}) &= \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in}) + (\mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in}) - \mathbf{f}_{out})t_{\mathbf{x}}(\mathbf{x}_{in}) \\ &= \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in}) + \frac{(\mathbf{f}_{out} - \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in})\mathbf{f}_{in})h_{\mathbf{x}}(\mathbf{x}_{in})}{h_{\mathbf{x}}(\mathbf{x}_{in})\mathbf{f}_{in}},\end{aligned}\quad (6)$$

where $\mathbf{f}_{in} = \mathbf{f}_1(\mathbf{x}_{in})$ and $\mathbf{f}_{out} = \mathbf{f}_2(\mathbf{x}_{out})$. In the case where h is explicitly time-dependent this becomes

$$\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{in}) = \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in}) + \frac{(\mathbf{f}_{out} - \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{in})\mathbf{f}_{in})h_{\mathbf{x}}(\mathbf{x}_{in}, t_{ref})}{h_t(\mathbf{x}_{in}, t_{ref}) + h_{\mathbf{x}}(\mathbf{x}_{in}, t_{ref})\mathbf{f}_{in}}. \quad (7)$$

In both cases we have that

$$\phi_{\mathbf{x}}(\mathbf{x}_0^{ref}, T) = \phi_{2,\mathbf{x}}(\mathbf{x}_{out}, T - t_{ref})\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{in})\phi_{1,\mathbf{x}}(\mathbf{x}_{in}, t_{ref}). \quad (8)$$

Introducing Noise

As in the deterministic case, we define the discontinuity boundary \mathcal{D} as the zeros of a function h . For a stochastically oscillating boundary we let h take the form

$$h(\mathbf{x}, t) = \hat{h}(\mathbf{x}, t) - P(t), \quad (9)$$

where the function \hat{h} is deterministic and $P(t)$ is a stochastic process.

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Let \hat{t}_{ref} be the time of flight from $\mathbf{x}_0^{\text{ref}}$ to the boundary in the absence of noise, i.e.

$$\hat{h}(\phi_1(\mathbf{x}_0^{\text{ref}}, \hat{t}_{\text{ref}})) = 0. \quad (10)$$

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We define Δt_{ref} to be the random variable given by the difference between \hat{t}_{ref} and the actual time of flight

$$\Delta t_{\text{ref}} = t_{\text{ref}} - \hat{t}_{\text{ref}}. \quad (11)$$

Stochastic Saltation Matrix

In order to deal with stochastically oscillating boundaries we extend the state space, such that the state vector and vector field are given by

$$\tilde{\mathbf{x}} = (\mathbf{x}, t, \Delta t_{\text{ref}})^T \quad \text{and} \quad \tilde{\mathbf{f}} = (\mathbf{f}, 1, 0)^T, \quad (12)$$

respectively.

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respectively.

We calculate the saltation matrix in this extended state space before projecting back. As a result, in the original state space we find that

$$\phi(\mathbf{x}_0, t) - \phi(\hat{\mathbf{x}}_0^{\text{ref}}, t) \approx \phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t)(\mathbf{x}_0 - \hat{\mathbf{x}}_0^{\text{ref}}) + \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})(\hat{\mathbf{f}}_{\text{in}} - \hat{\mathbf{f}}_{\text{out}})\Delta t_{\text{ref}}, \quad (13)$$

where

$$\phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t) = \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})\mathbf{D}_{\mathbf{x}}^*(\hat{\mathbf{x}}_{\text{in}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}}) \quad (14)$$

and

$$\mathbf{D}_{\mathbf{x}}^*(\hat{\mathbf{x}}_{\text{in}}) = \mathbf{I} + \frac{(\hat{\mathbf{f}}_{\text{out}} - \hat{\mathbf{f}}_{\text{in}})\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}})}{\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}})\hat{\mathbf{f}}_{\text{in}} + \hat{h}_t(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}}) - V(\hat{t}_{\text{ref}}|P(\hat{t}_{\text{ref}}) = 0)}. \quad (15)$$

In all the above $\hat{\cdot}$ indicates the values associated with the deterministic reference trajectory.

Summary

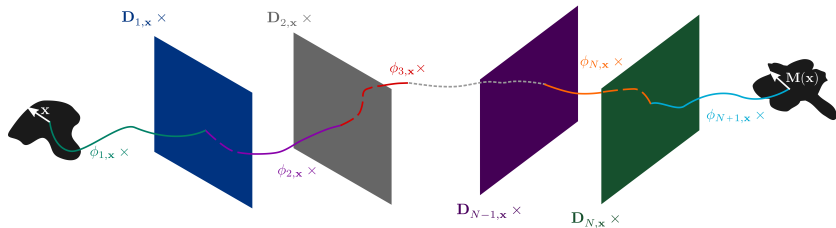


Figure: Linearising Discontinuous Systems

Summary

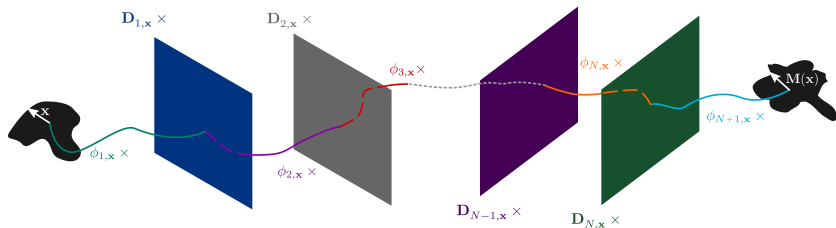
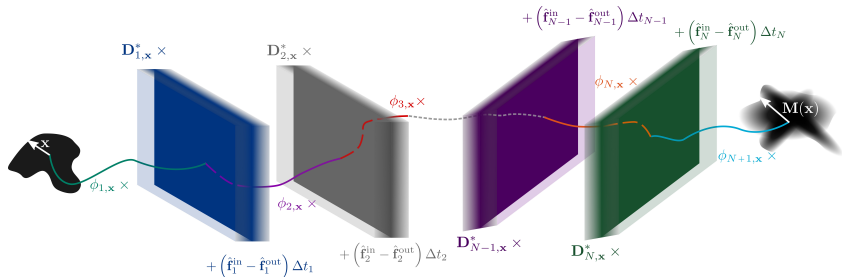


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The Chua Circuit

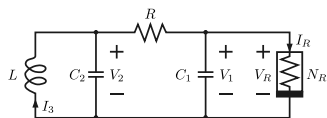


Figure: The Chua Circuit

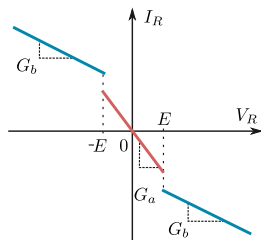


Figure: The $V - I$ characteristic of the Chua Diode.

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presence of chaos was shown experimentally, numerically and mathematically [CKM86].
- Contains four linear elements and one nonlinear resistor known as a *Chua diode*.
- Easily and cheaply constructed using standard electronic components [Ken92].

System equations

The dynamics of the Chua circuit can be described by the following nondimensionalised state equations

$$\begin{aligned}\frac{dx}{dt} &= \alpha(y - x - g(x)), \\ \frac{dy}{dt} &= x - y + z, \\ \frac{dz}{dt} &= -(\beta y + \gamma z),\end{aligned}\tag{16}$$

where $g(x)$ is the piecewise linear function representing the V - I characteristic of Chua's diode

$$g(x) = \begin{cases} m_1 x + m_1 - m_0 & \text{if } x < -1, \\ (m_0 - \epsilon)x & \text{if } |x| \leq 1, \\ m_1 x + m_0 - m_1 & \text{if } x > 1. \end{cases}\tag{17}$$

Complicated Dynamics

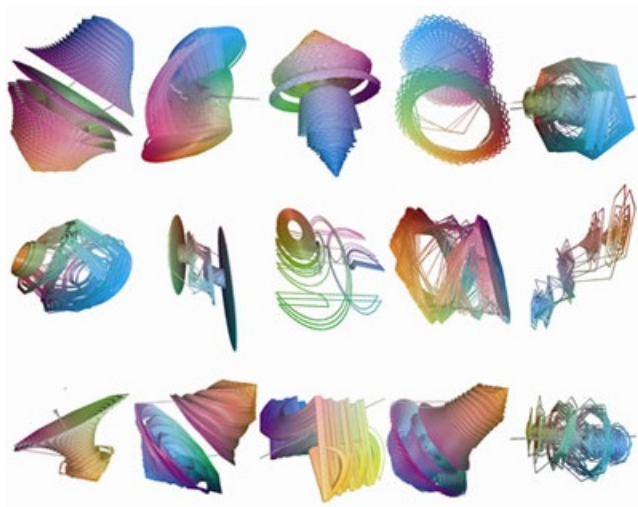


Figure: A Zoo of Attractors Produced by the Chua Circuit [BP08]

Hidden and Self-Excited Attractors

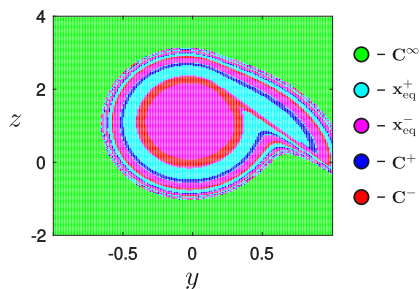
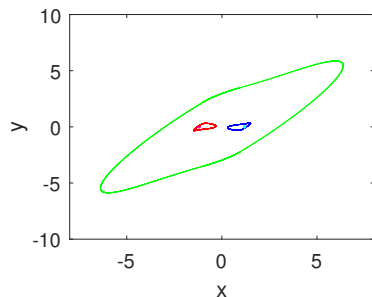
Hidden attractors: have basins of attraction that do not intersect with small neighborhoods of equilibria.

Self-excited attractors: Can be found by following trajectories from the neighbourhoods of unstable equilibria until the end of a transient process [LK13].

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For a range of parameter values the Chua circuit has a 5-stable regime including 3 hidden periodic attractors [SKLC17].

A Discontinuous Model

Provided the magnitude of ϵ is not too large the hidden attractors in the 5-stable regime continue to exist and can be easily found by numerical continuation.

They are destroyed in saddle-bifurcations if the magnitude of ϵ grows too large.

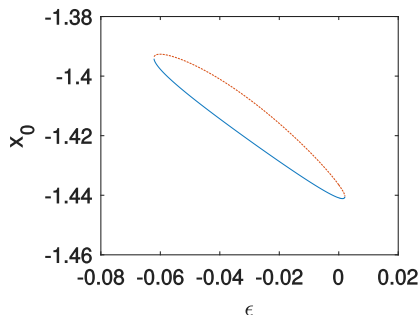


Figure: Bifurcation diagram showing the saddle bifurcations of C^- as the magnitude of ϵ grows. Here $\alpha = 8.4$, $\beta = 12$, $\gamma = -0.005$, $m_0 = -1.2$ and $m_1 = 0.145$.

Steady-State Distributions

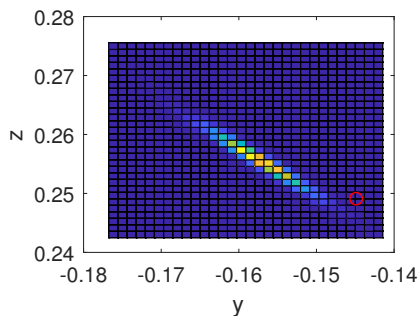


Figure: Steady state distribution of orbit errors on the discontinuity boundary \mathcal{D}^- for trajectories with initial condition on the periodic orbit \mathcal{C}^- .

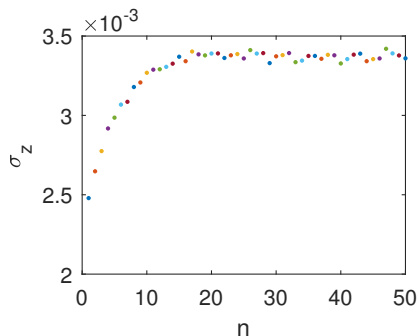
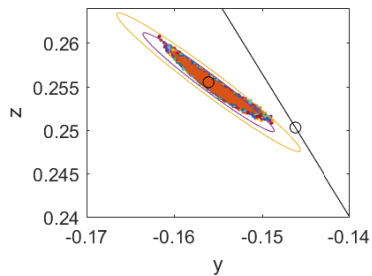
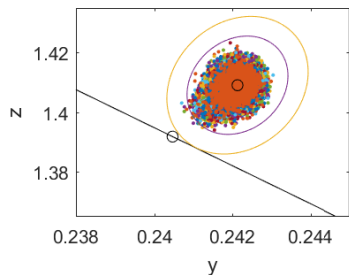
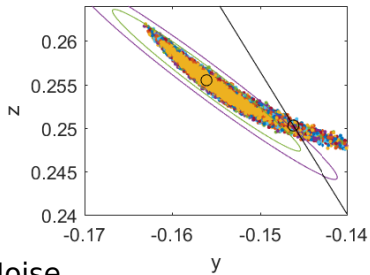
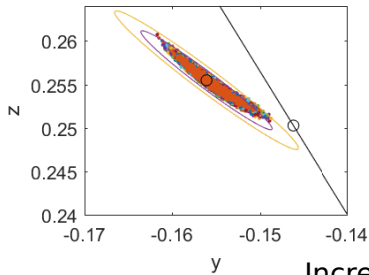
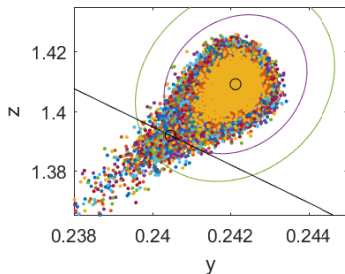
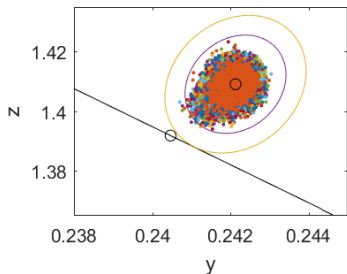


Figure: Convergence of σ_z to its steady state value for the distribution shown on the left.

Destroying Periodic Attractors



Destroying Periodic Attractors

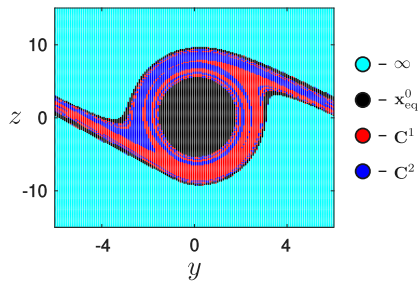
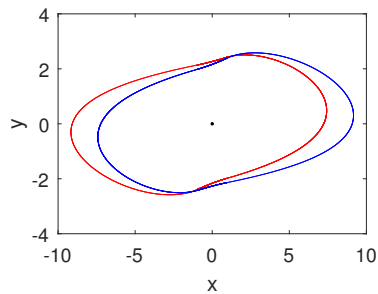


Increasing Noise



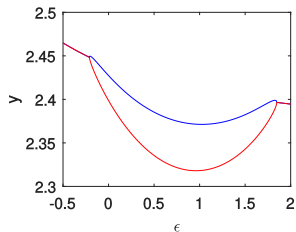
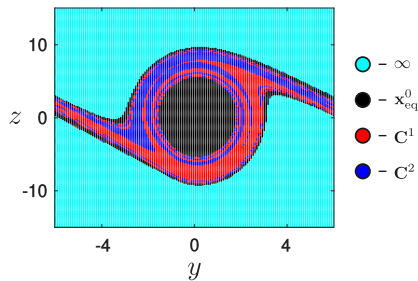
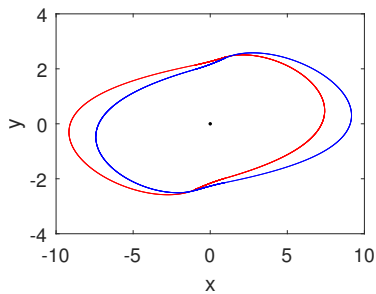
A 3-Stable Regime

For a range of parameter values the Chua circuit has a 3-stable regime including 2 symmetric periodic attractors.



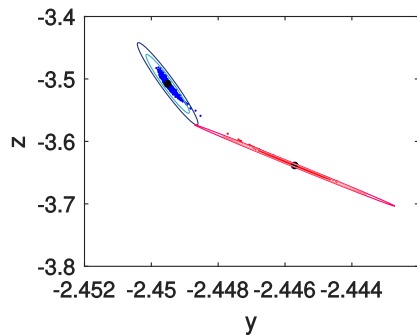
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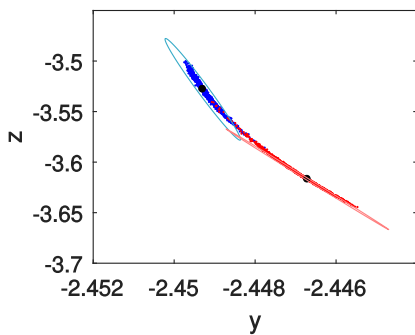
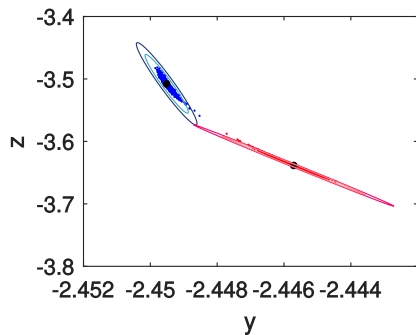


The two periodic attractors merge in a supercritical pitchfork bifurcation if the magnitude ϵ grows too large.

Merging Periodic Attractors



Merging Periodic Attractors



Towards Bifurcation



Conclusions






- The concept of a saltation matrix can be generalised to stochastic systems.

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- This can be used to estimate the dynamics of discontinuous systems with noisy boundaries.
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 - multi/monostability
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Conclusions

- The concept of a saltation matrix can be generalised to stochastic systems.
- This can be used to estimate the dynamics of discontinuous systems with noisy boundaries.
 - destruction of attractors
 - multi/monostability
 - merging of attractors
 - flickering/switching
- It remains to generalise our method
 - Higher order terms for continuous systems
 - Non-identity boundary mappings
 - Dealing with non-transversal intersections
 - Rugged, stochastic discontinuity surfaces

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