Noise and Multistability in the Square Root Map

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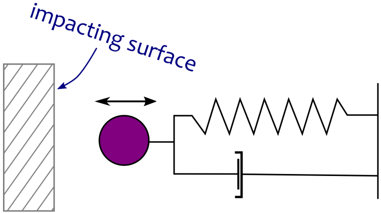
The Square Root Map

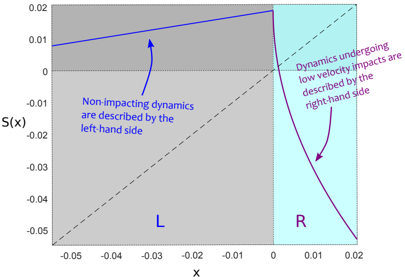
Many impacting systems are described by a 1-D map known as the square root map near *grazing* impacts.

$$x_{n+1} = S(x_n) = \begin{cases} \mu + b x_n & \text{if } x_n < 0, \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0, \end{cases}$$

where $a > 0$ and $b > 0$.



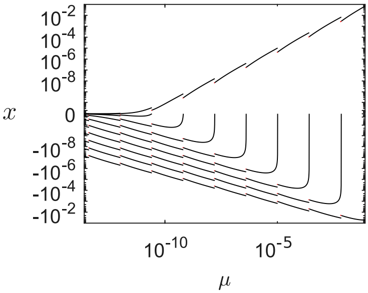
A forced impact oscillator



Symbolically, if $x_n < 0$ it is represented by an L and if $x_n > 0$ it is represented by an R.

The Period Adding Cascade

Here we will assume that the parameter b (the slope of the linear part) is such that $0 < b < 1/4$. For values of b in this range the deterministic square root map undergoes a period-adding cascade with intervals of multistability as the bifurcation parameter μ is decreased.



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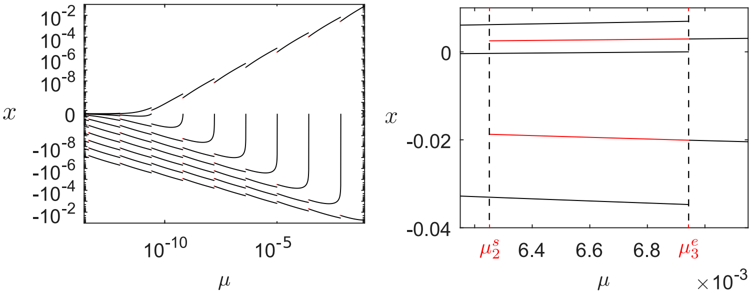

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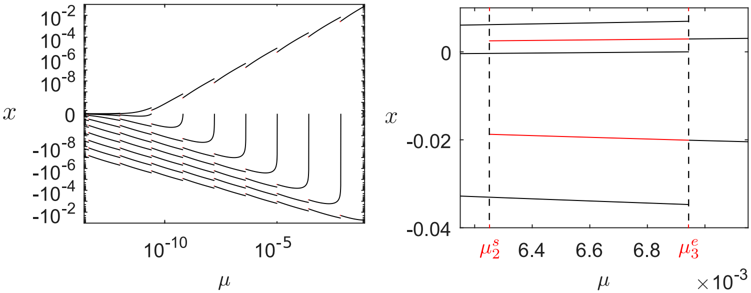

These periodic orbits take the form $(RL^m)^\infty$ for $m = 1, 2, 3, \ldots$. This means they consist of one iterate on the right (> 0) followed by m iterates on the left (< 0).

Riddled Basins of Attraction

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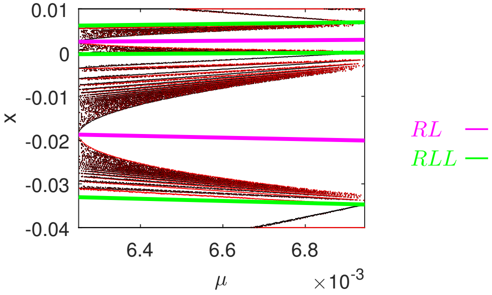

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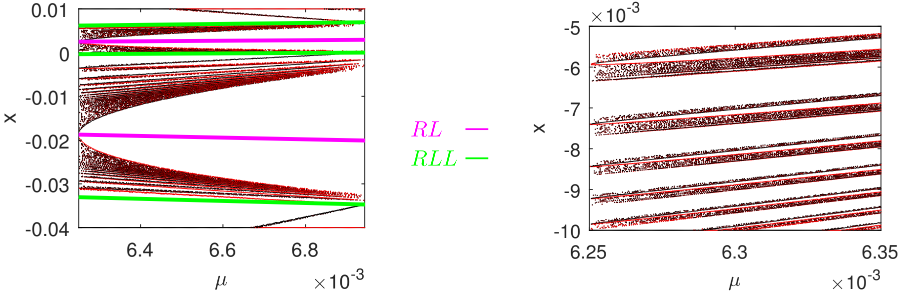

The Square Root Map With Additive Noise

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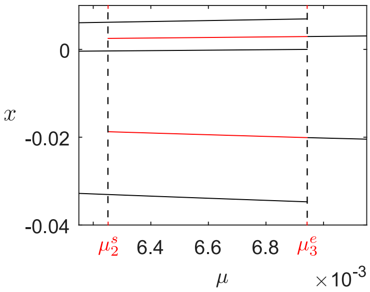
The square root map with additive Gaussian white noise is given by

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0, \end{cases} \tag{1}$$

where ξ_n are identically distributed independent normal random variables with mean 0 and standard deviation Δ, $\xi_n \sim N(0, \Delta^2)$.

Noisy Bifurcation Diagrams



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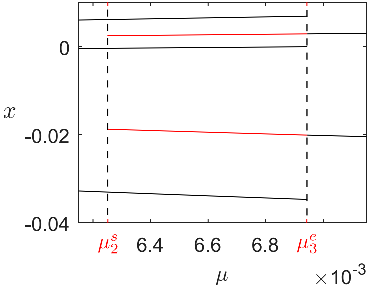

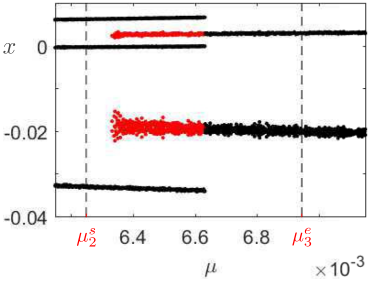

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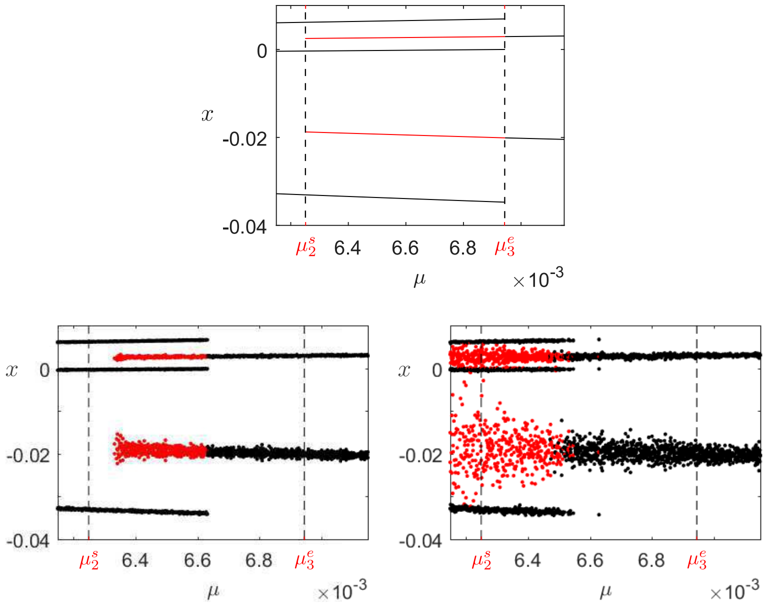
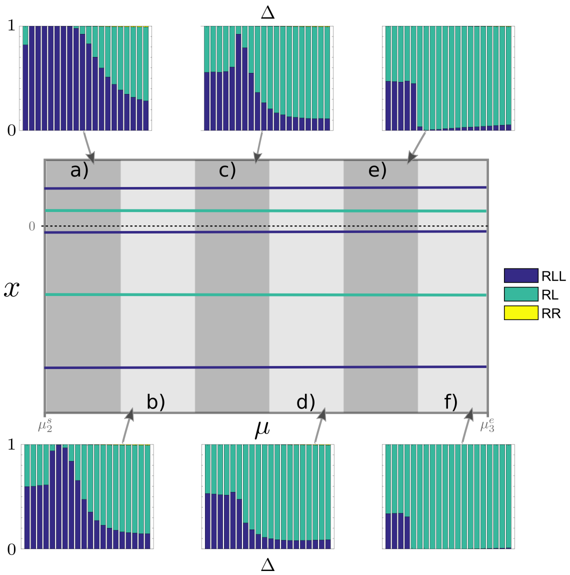


Noise Amplitude and Proportions of Periodic Behaviour



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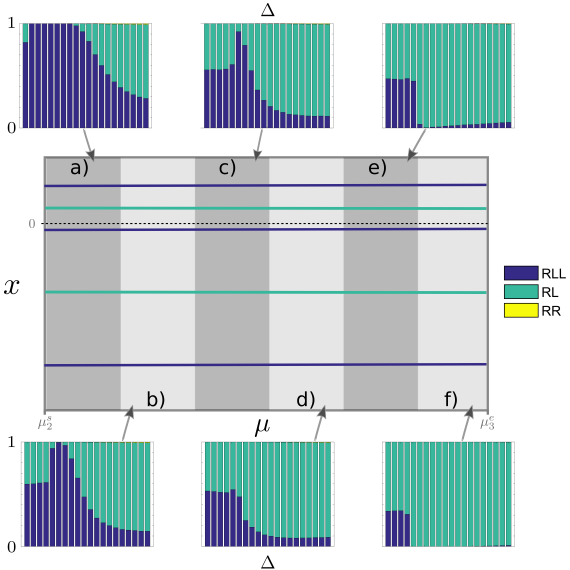

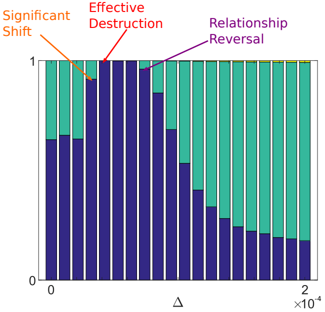

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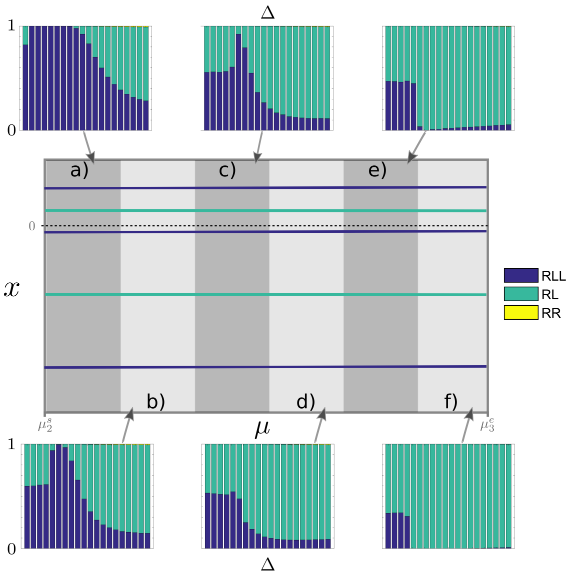
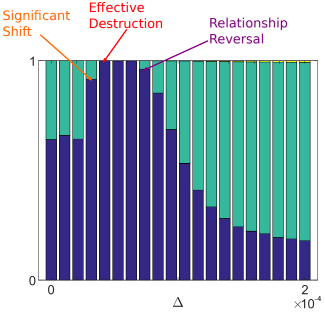
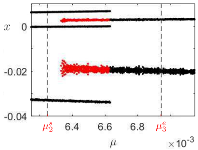
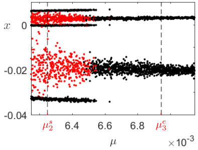
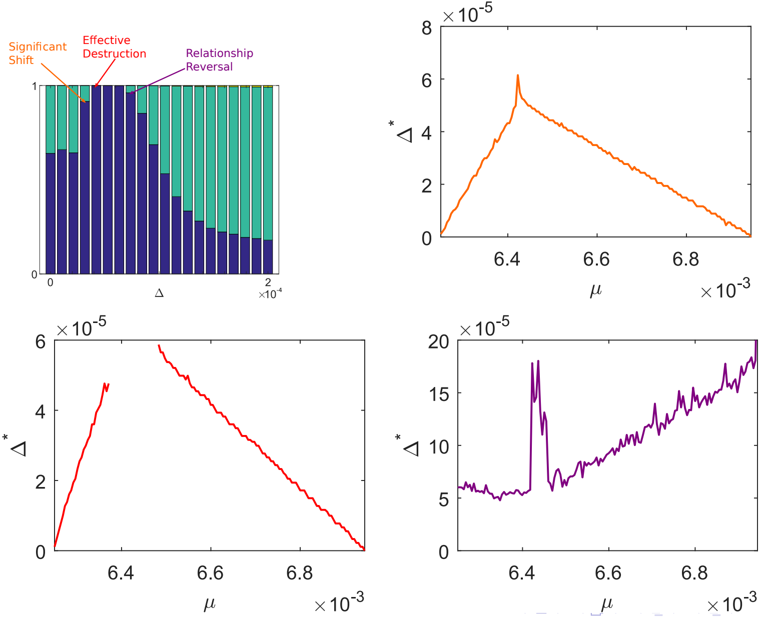






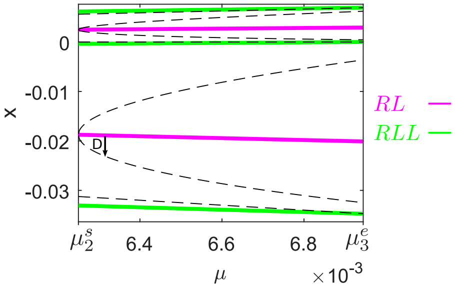


Threshold Noise Amplitudes



Basins and Steady State Distributions

Primary basins:



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Basins and Steady State Distributions

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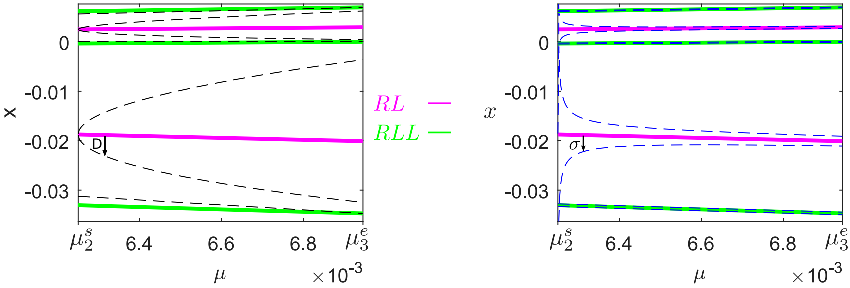
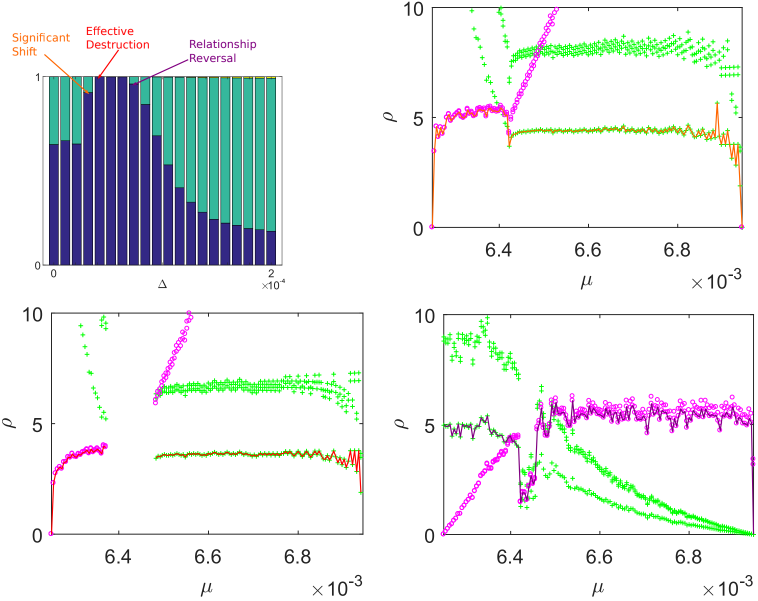


We consider threshold values of $\rho = D/\sigma$. ρ gives us some measure of how likely it is for noise to push the dynamics out of the basin of attraction.

Threshold ρ Values



Scaling

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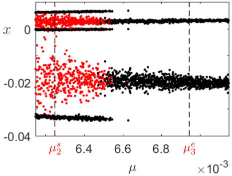
We find that choosing the noise amplitude to be $\Delta' = b^2\Delta$ on the interval of multistability $(\mu^s_{m+1}, \mu^e_{m+2})$ will result in a similar effect of noise on the dynamics of the map as choosing the noise amplitude to be Δ on the interval (μ^s_m, μ^e_{m+1}) for large m.

Inducing Multistability

We have previously seen that noise of an appropriate amplitude also has the potential to induce multistability in regions close to, but outside, intervals of multistability.



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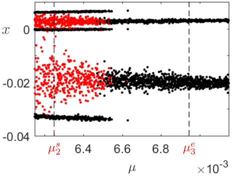

We have found that noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

$$RLLRLL \dots RLL\underline{RLRRL}RL \dots RLRL. \quad (2)$$

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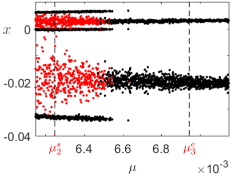

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$$RLLRLL\ldots RLL\underline{RLRRL}RL\ldots RLRL. \tag{2}$$

The significant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map.

Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

$$A_{RR} = \left(0, (\mu/a)^2\right). \tag{3}$$

We also note that the last left iterate of the period-3 orbit is very close to 0 for values of μ close to the interval of multistability.

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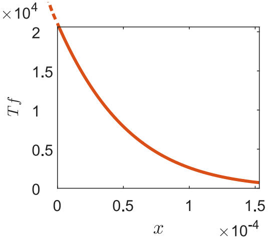
Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into A_{RR}, inducing repeated R's or repeated low-velocity impacts.

Noise and Deterministic Structures

RL —
RLL —



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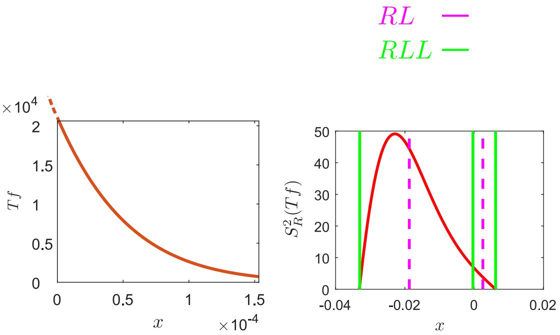

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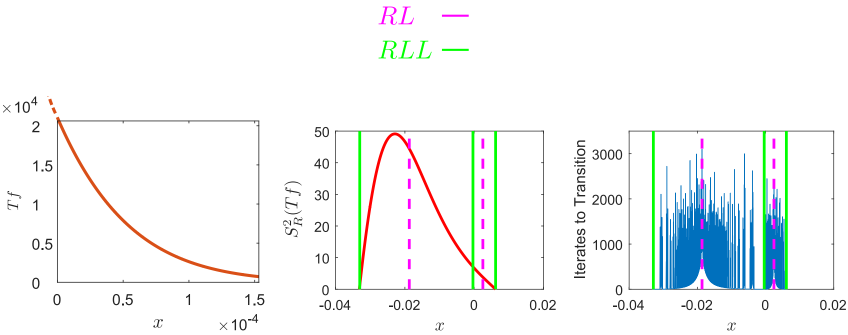

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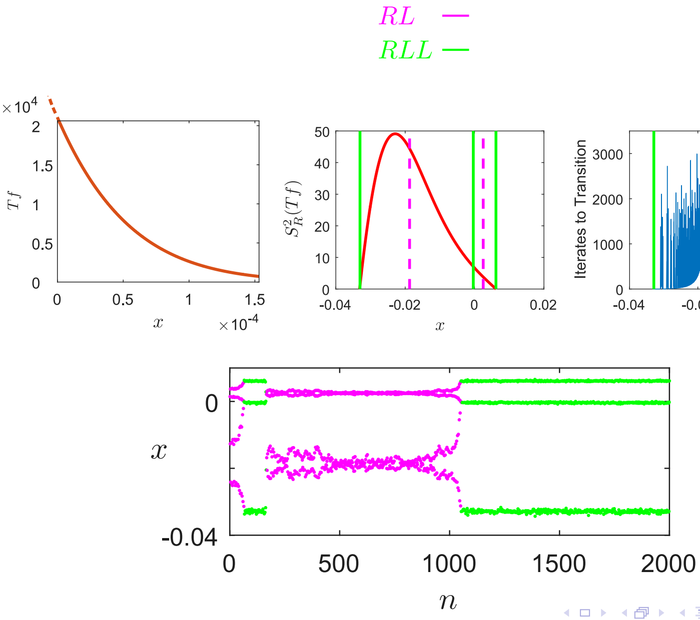

Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from RL^m behaviour to RL^{m-1} behaviour for increasing m. In particular we observe transitions of the form

$$RL^m RL^m \ldots RL^m \underline{RL^{m-1} RL^{k-2} RL^{m-1}} RL^{m-1} \ldots \ldots RL^{m-1} \tag{4}$$

for μ in a neighbourhood of μ_m^s such that $\mu < \mu_m^s$ and $k \in \{2, 3, \ldots, m\}$.

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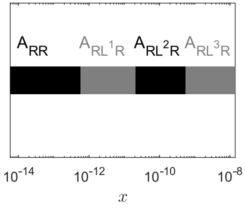

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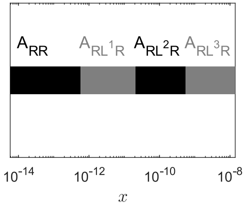

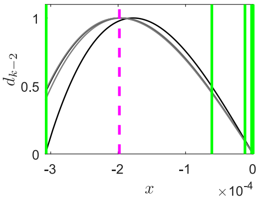

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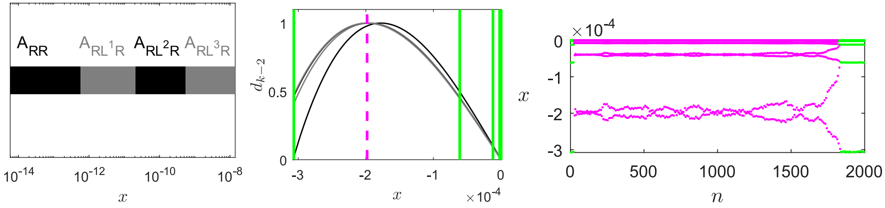

Conclusions

- Additive noise has a complex nonmonotonic effect on the proportion of iterates spent in coexisting periodic behaviours on intervals of multistability.
- The relationship observed is highly dependent on the value of the bifurcation parameter μ.
- We can explain these relationships by examining how the steady-state distributions associated with periodic orbits interact with their basins of attraction.
- The effect of the addition of noise on intervals of multistability of increasing minimal periodic orbit obeys a scaling law.
- Additive noise has the potential to induce multistability outside such intervals.

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