Noise and Multistability in the Square Root Map

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10 July 2018

Noise and Nonsmoothness in Dynamical Systems

Both noise and nonsmoothness have been shown to independently be the drivers of significant changes in qualitative behaviour.

- Nonsmooth systems - qualitative changes in the behavior of the system under parameter variation that do not occur in the smooth setting.
- Adding noise to (smooth) systems - does more than just blur the outcome of the system in the absence of noise

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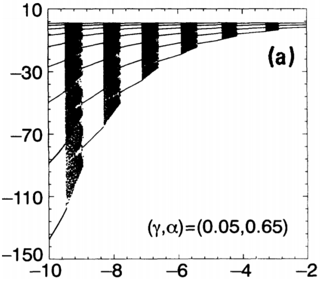

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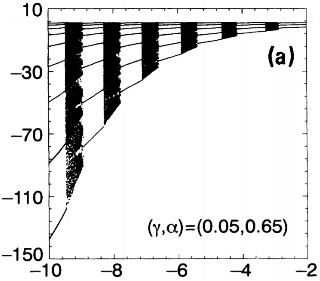


Figure: From [CONG94].

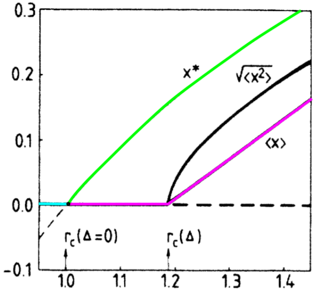


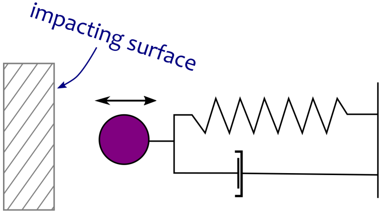
Figure: Adapted from [LL86].

The Square Root Map

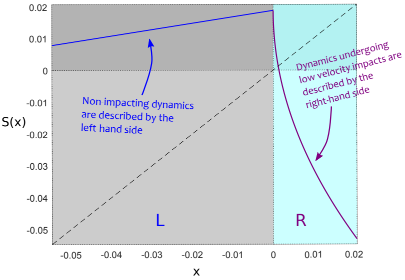
Many impacting systems, including rattling gears, moored boats impacting docks, Braille printers, percussive drilling and atomic force microscopes are described by a 1-D map known as the square root map near *grazing* impacts.

$$x_{n+1} = S(x_n) = \begin{cases} \mu + bx_n & \text{if } x_n < 0, \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0, \end{cases}$$

where $a > 0$ and $b > 0$.

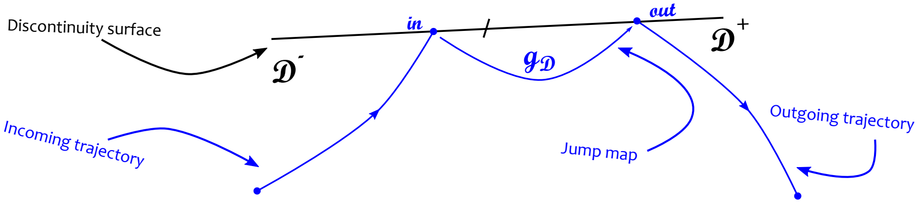


A forced impact oscillator



Symbolically, if $x_n < 0$ it is represented by an L and if $x_n > 0$ it is represented by an R.

Deriving The Square Root Map



Deriving the map from the full system

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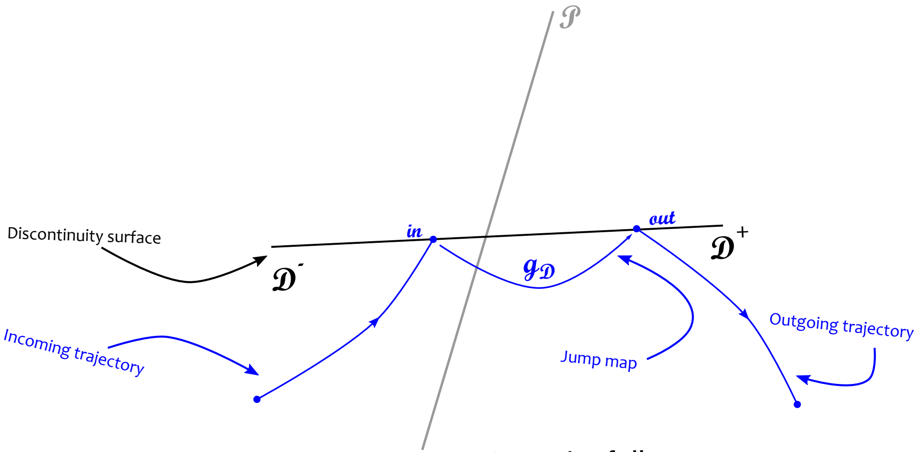

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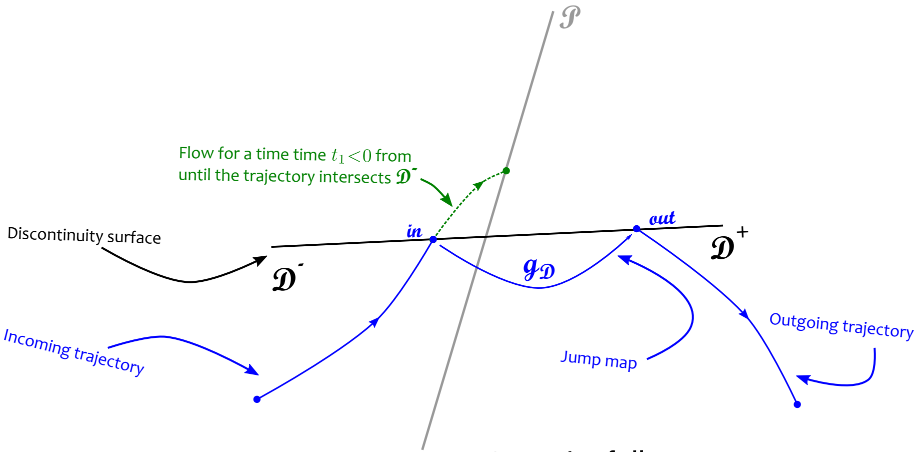

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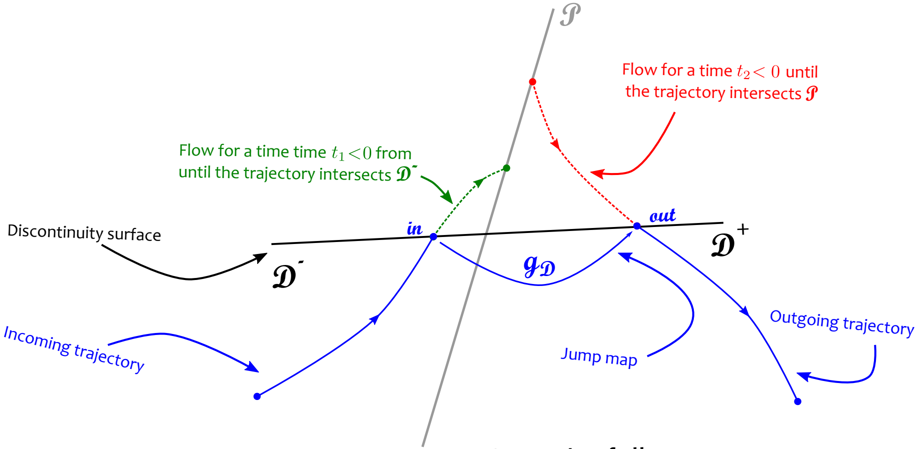

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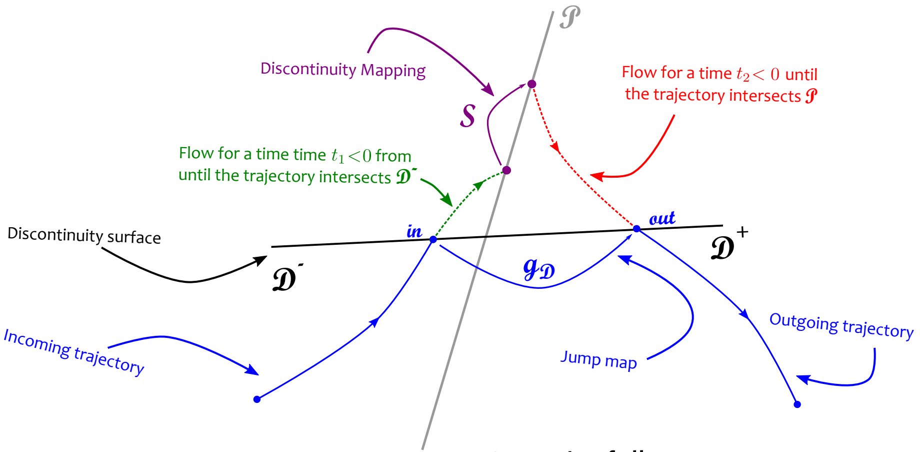
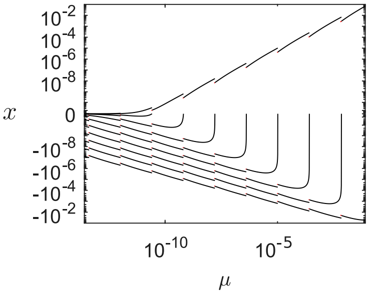


Deriving the map from the full system

The Period Adding Cascade

Here we will assume that the parameter b (the slope of the linear part) is such that $0 < b < 1/4$. For values of b in this range the deterministic square root map undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter μ is decreased.



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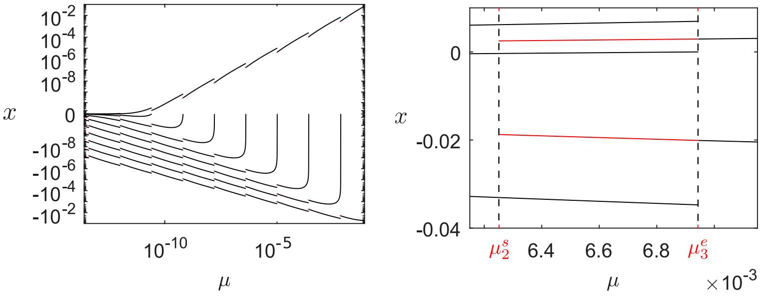

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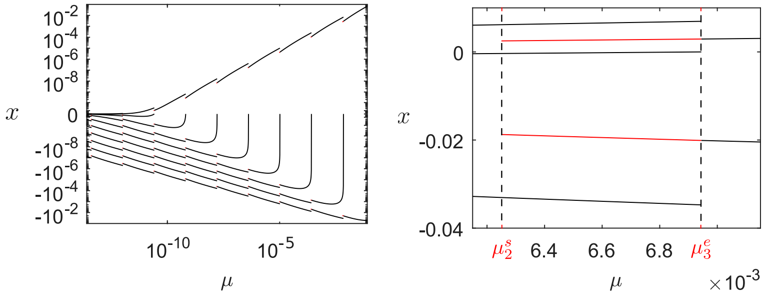

These periodic orbits take the form $(RL^m)^\infty$ for $m = 1, 2, 3, \ldots$. This means they consist of one iterate on the right (> 0) followed by m iterates on the left (< 0).

Riddled Basins of Attraction

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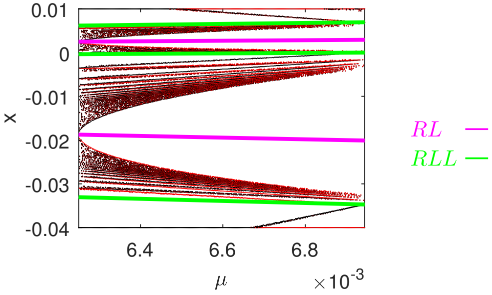

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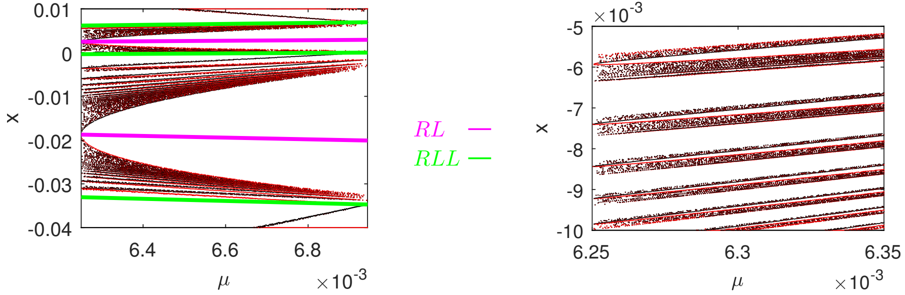

The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that white noise in the piecewise smooth flow translates to additive white noise in the square root map. This noise formulation may be sensible to model systems where the forcing term or external fluctuations represent a significant source of uncertainty.

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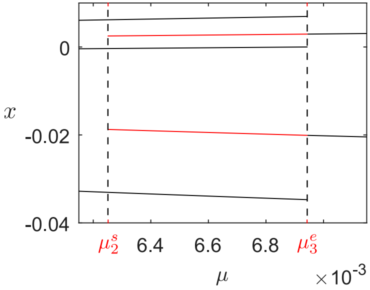
The square root map with additive Gaussian white noise is given by

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0, \end{cases} \tag{1}$$

where ξ_n are identically distributed independent normal random variables with mean 0 and standard deviation Δ, $\xi_n \sim N(0, \Delta^2)$.

Noisy Bifurcation Diagrams



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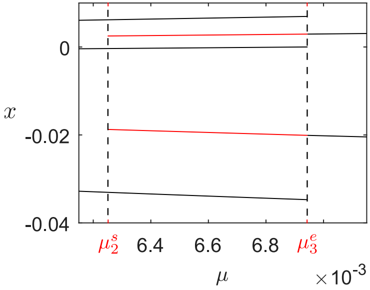

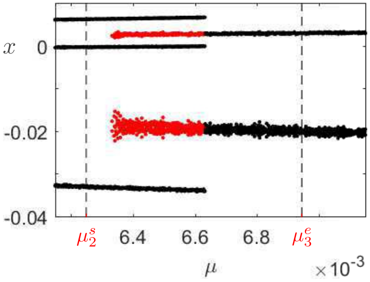

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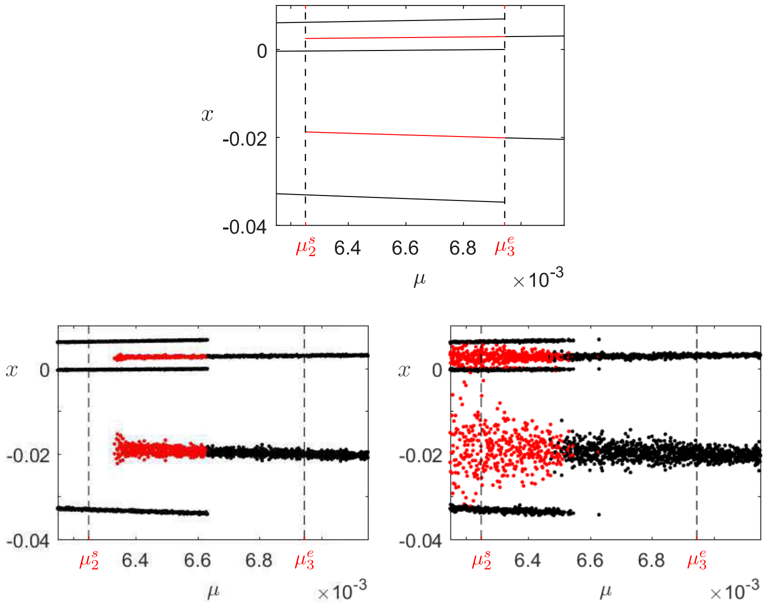
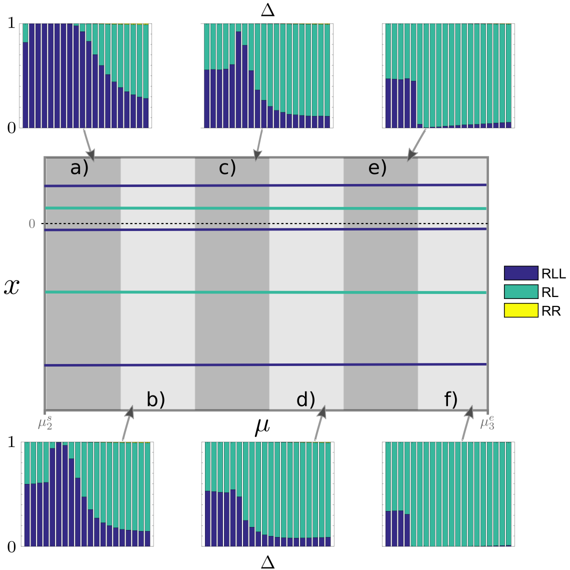


Noise Amplitude and Proportions of Periodic Behaviour



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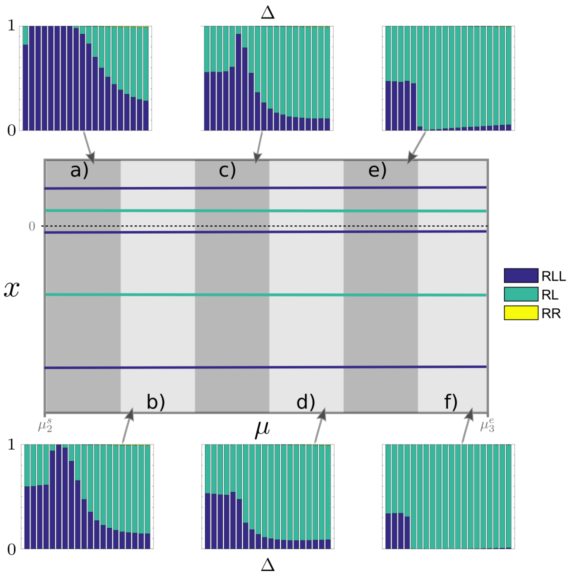

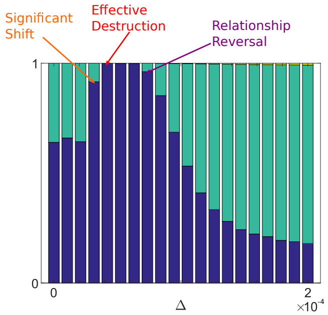

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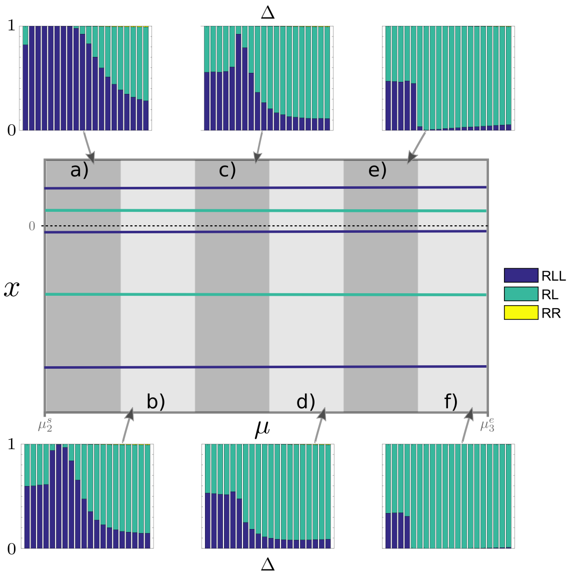
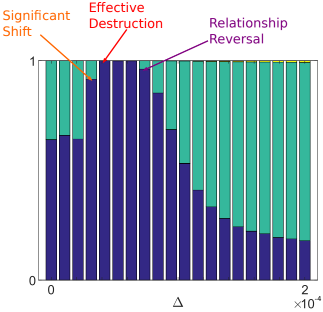
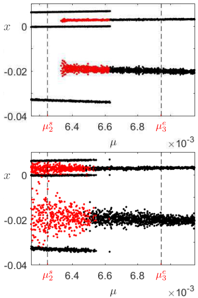
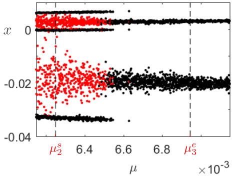






Inducing Bistability

We have previously seen that noise of an appropriate amplitude also has the potential to induce bistability in regions close to, but outside, intervals of bistability.



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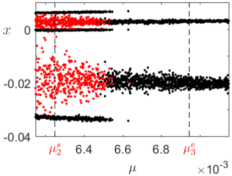

In the numerical simulations we have found that noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

$$RLLRLL\ldots RLL\underline{RLRRL}RL\ldots RLRL. \tag{2}$$

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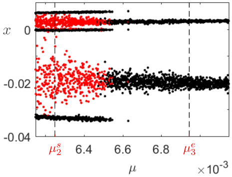

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$$RLLRLL \ldots RLL\underline{RLRRL}RL \ldots RLRL. \tag{2}$$

The significant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map, i.e. repeated low-velocity impacts in the physical system.

Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

$$A_{RR} = \left(0, (\mu/a)^2\right). \tag{3}$$

We also note that the last left iterate of the period-3 orbit is very close to 0 for values of μ close to the interval of multistability.

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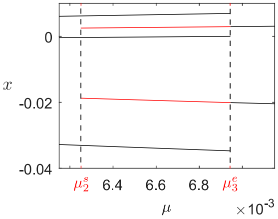
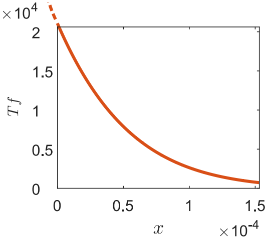


Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into A_{RR} inducing repeated R's or repeated grazing impacts.

Noise and Deterministic Structures

RL —

RLL —



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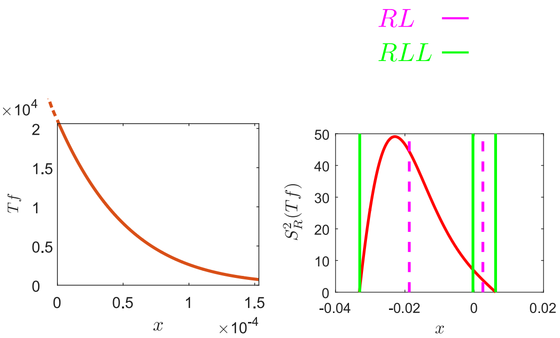

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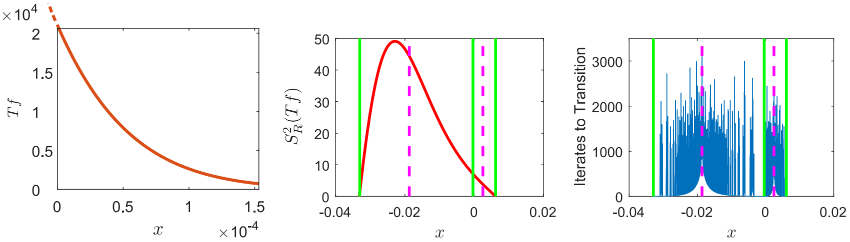

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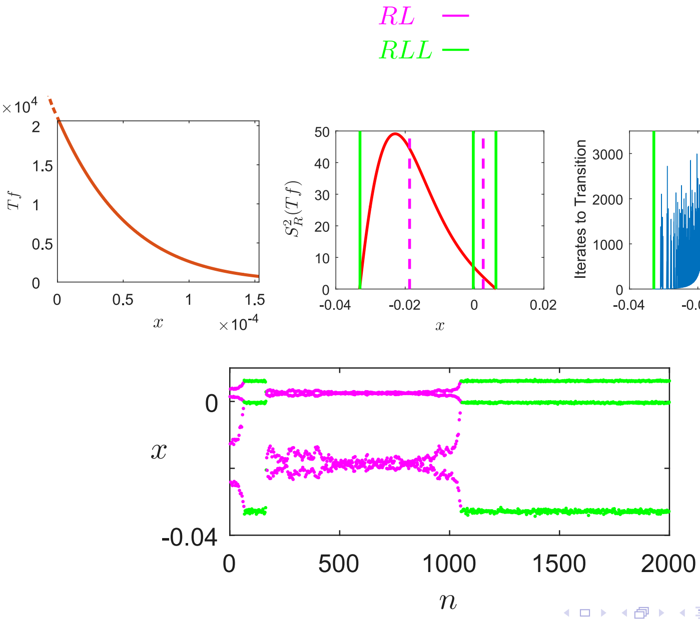

Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from RL^m behaviour to RL^{m-1} behaviour for increasing m. In particular we observe transitions of the form

$$RL^m RL^m \dots RL^m \underline{RL^{m-1} RL^{k-2} RL^{m-1}} RL^{m-1} \dots\dots RL^{m-1} \tag{4}$$

for μ in a neighbourhood of μ_m^s such that $\mu < \mu_m^s$ and $k \in \{2, 3, \dots, m\}$.

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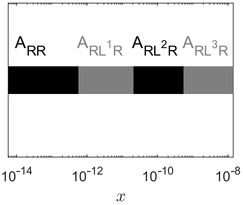

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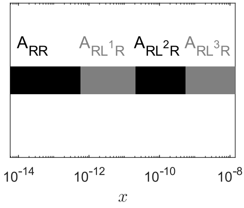

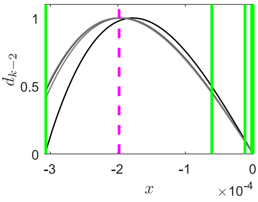

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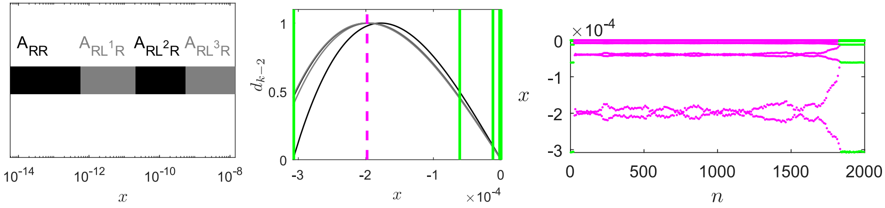

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- This behaviour can be generalised to higher periodicities.
- The effect of the addition of noise on intervals of multistability of increasing minimal periodic orbit obeys a scaling law.

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