Linearising Piecewise-Smooth Flows with Stochastic Discontinuity Boundaries

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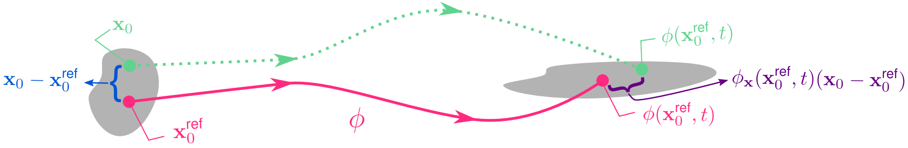
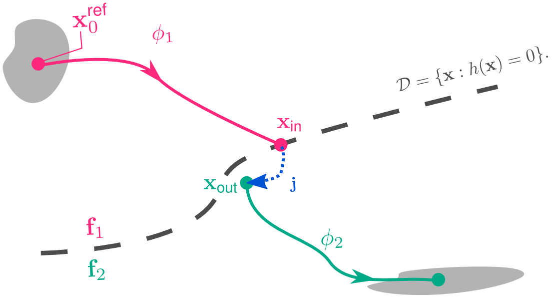
Linearisation

In a smooth dynamical system the characteristics of a given reference trajectory can be determined by examining the linearised system about the reference trajectory.



This form of analysis cannot be used in nonsmooth systems as $\mathbf{f}$ is not everywhere differentiable, or $\phi(\mathbf{x}_0^{\text{ref}}, t)$ is not continuous.

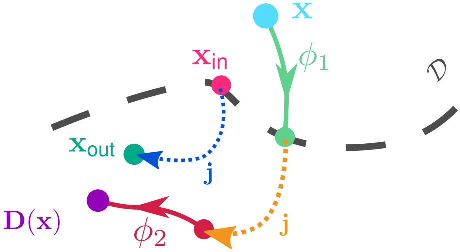


The Zero-Time Discontinuity Mapping

To account for this we derive the zero-time discontinuity mapping (ZDM) $\mathbf{D}$, such that

$$\phi(\mathbf{x}_0, t) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), t - t_{\text{ref}}), \tag{1}$$

where t_{ref} is the time of flight of the reference trajectory to the discontinuity boundary.



$\mathbf{D}$ is given by

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x})), \tag{2}$$

where $t(\mathbf{x})$ is the (possibly negative) time of flight from $\mathbf{x}$ to the boundary.

The Saltation Matrix

In this talk we will consider the simplified case where the jump mapping $\mathbf{j}$ is the identity mapping. In this case the Jacobian derivative of $\mathbf{D}$ evaluated at $\mathbf{x}_{\text{in}}$ is given by

$$\begin{aligned} \mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) &= \mathbf{I} + (\mathbf{f}_{\text{in}} - \mathbf{f}_{\text{out}})t_{\mathbf{x}}(\mathbf{x}_{\text{in}}) \\ &= \mathbf{I} + \frac{(\mathbf{f}_{\text{out}} - \mathbf{f}_{\text{in}})h_{\mathbf{x}}(\mathbf{x}_{\text{in}})}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}}, \end{aligned} \tag{3}$$

where $\mathbf{f}_{\text{in}} = \mathbf{f}_1(\mathbf{x}_{\text{in}})$ and $\mathbf{f}_{\text{out}} = \mathbf{f}_2(\mathbf{x}_{\text{out}})$. In the case where h is explicitly time-dependent this becomes

$$\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) = \mathbf{I} + \frac{(\mathbf{f}_{\text{out}} - \mathbf{f}_{\text{in}})h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})}{h_t(\mathbf{x}_{\text{in}}, t_{\text{ref}}) + h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}}}. \tag{4}$$

In both cases this matrix allows us to compose the Jacobians of the individual flows to give the overall Jacobian

$$\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, T) = \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, T - t_{\text{ref}})\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\phi_{1,\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}}). \tag{5}$$

Adding Noise to the Boundary

In order to deal with a stochastically moving boundary we repeat the analysis in an extended state space, such that the state vector is $\tilde{\mathbf{x}} = (\mathbf{x}, t, \Delta t_{\text{ref}})^T$. Here Δt_{ref} is the random quantity which represents the difference in the hitting time of the reference trajectory in the stochastic system compared to the corresponding deterministic system.

We calculate the saltation matrix in this extended state space before projecting back. As a result, in the original state space we find that

$$\phi(\mathbf{x}_0, t) - \phi(\hat{\mathbf{x}}_0^{\text{ref}}, t) \approx \phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t)(\mathbf{x}_0 - \hat{\mathbf{x}}_0^{\text{ref}}) + \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})(\hat{\mathbf{f}}_{\text{in}} - \hat{\mathbf{f}}_{\text{out}})\Delta t_{\text{ref}}, \tag{6}$$

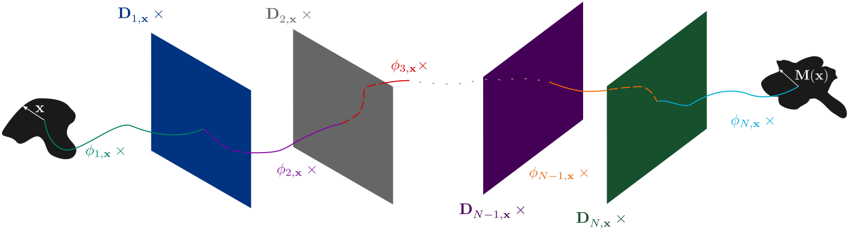
where

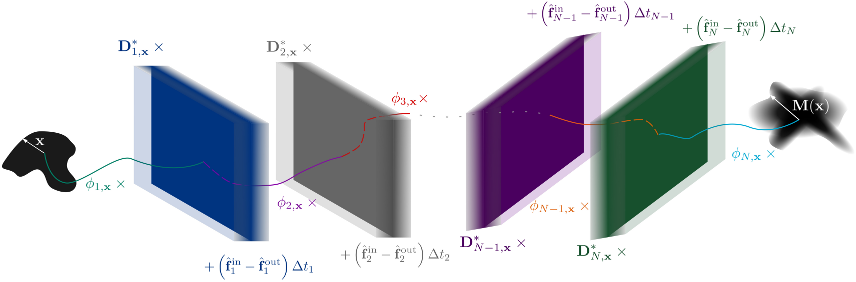
$$\phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t) = \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})\mathbf{D}^*_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}}) \tag{7}$$

and

$$\mathbf{D}^*_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}) = \mathbf{I} + \frac{(\hat{\mathbf{f}}_{\text{out}} - \hat{\mathbf{f}}_{\text{in}})\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})}{\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})\hat{\mathbf{f}}_{\text{in}} - \hat{v}(\hat{t}_{\text{ref}}) - V(\hat{t}_{\text{ref}} | P(\hat{t}_{\text{ref}}) = 0)} \tag{8}$$

A Comparison





A PWL Example

Consider the piecewise linear system in the plane given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{f_1}(\mathbf{x}) = \mathbf{A}_L\mathbf{x} + \mathbf{u}_L \text{ if } \mathbf{x} \in S^-, \\ \mathbf{f_2}(\mathbf{x}) = \mathbf{A}_R\mathbf{x} + \mathbf{u}_R \text{ if } \mathbf{x} \in S^+, \end{cases} \tag{9}$$

where the two regions

$$S^- = \{\mathbf{x} = (x, y) \colon x < 0\} \quad \text{and} \quad S^+ = \{\mathbf{x} = (x, y) \colon x \geq 0\} \tag{10}$$

are separated by the discontinuity boundary

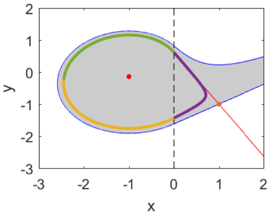
$$\mathcal{D} = \{\mathbf{x} = (x, y) \colon h(\mathbf{x}) = x = 0\}. \tag{11}$$

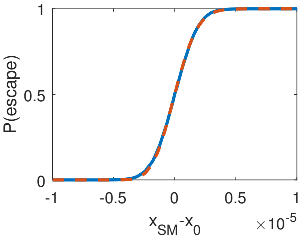
Take

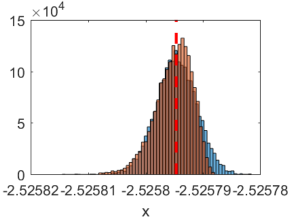
$$\mathbf{A}_L = \begin{pmatrix} 2\gamma & -1 \\ \gamma^2 + 1 & 0 \end{pmatrix}, \ \mathbf{u}_L = \begin{pmatrix} 0 \\ \gamma^2 + 1 \end{pmatrix}, \ \mathbf{A}_R = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{u}_R = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

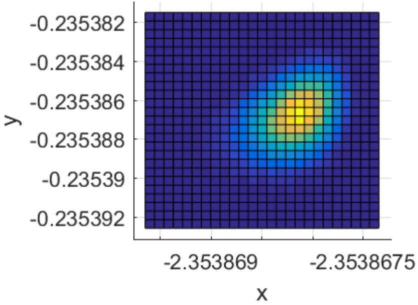
It has been shown by Ponce et al [PRV13] that in this system a stable limit cycle bifurcates as γ increases through 0 in a *focus-center-limit cycle bifurcation*. This limit cycle exists provided γ is sufficiently small.

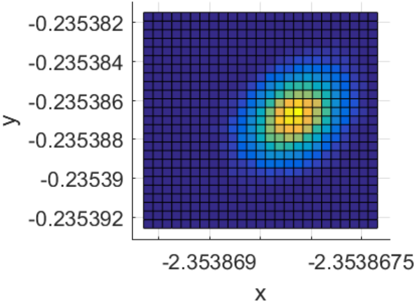
A PWL Example



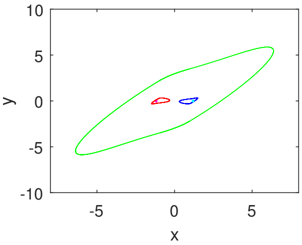




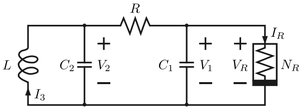




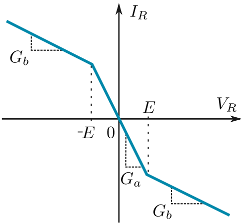
Future Work



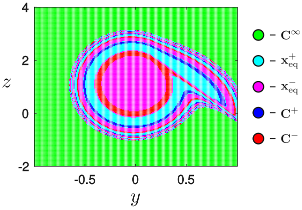
10
5
0
-5
-10
y
-5
0
5
x



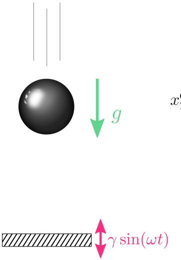
R
I_R
L
C_2
V_2
C_1
V_1
V_R
N_R
I_3



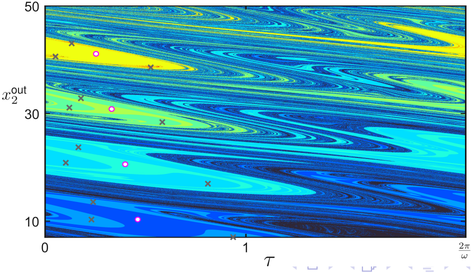
I_R
G_b
V_R
E
-E
0
G_a
G_b



4
2
0
-2
z
-0.5
0
0.5
y
– $\mathbf{C}^{\infty}$
– $\mathbf{x}_{\text{eq}}^{+}$
– $\mathbf{x}_{\text{eq}}^{-}$
– $\mathbf{C}^{+}$
– $\mathbf{C}^{-}$



g
$\gamma \sin(\omega t)$



50
30
10
x_2^{out}
0
1
$\frac{2\pi}{\omega}$
τ

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