ASBG-Colourings of Trees

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An ordering of the vertices is allowable if the vertices of each part can be embedded in that order on two parallel lines in the plane such that the edges incident with each vertex alternate in colour (beginning and ending with blue) in that embedding.

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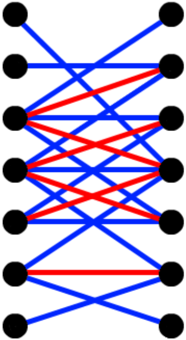

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We call a colouring of a graph that satisfies the above conditions a *feasible colouring*.

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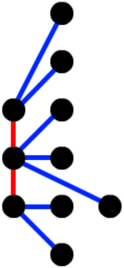

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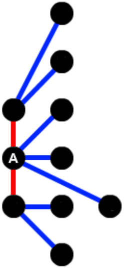

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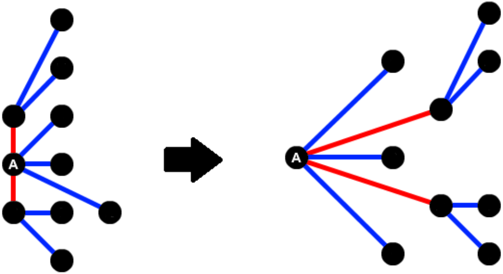

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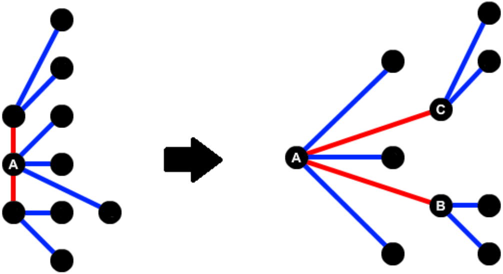

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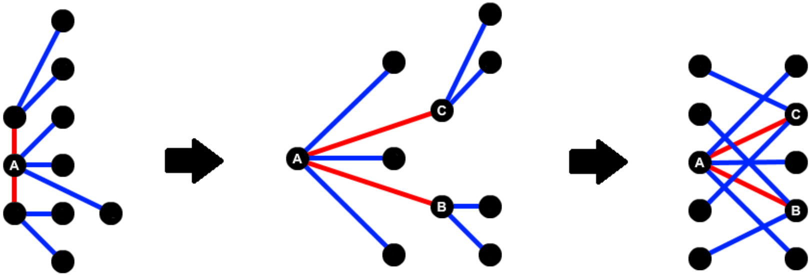

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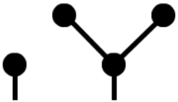

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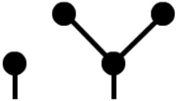

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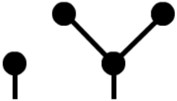

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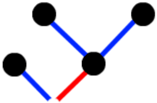

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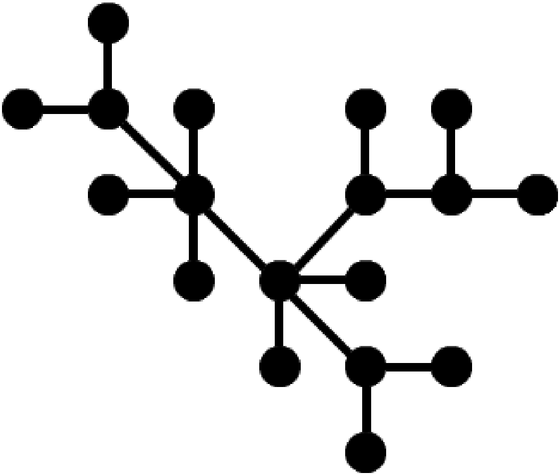
Corollary: If a tree T has an ASBG-colouring c, then c is the unique ASBG-colouring of T.

Example



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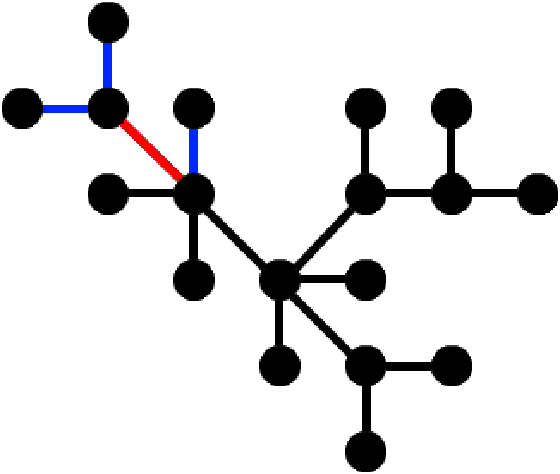

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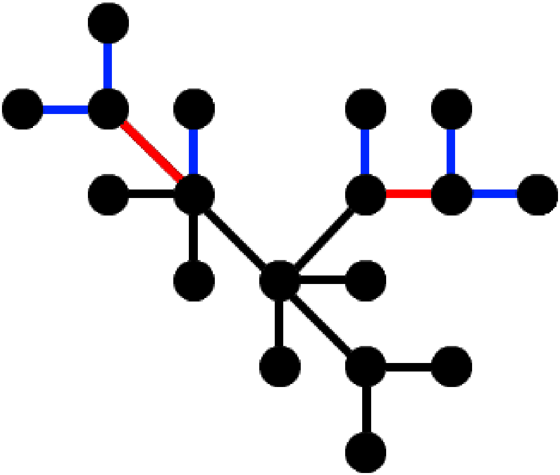

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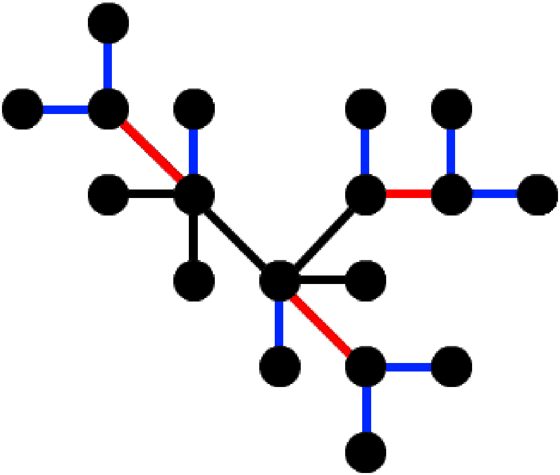

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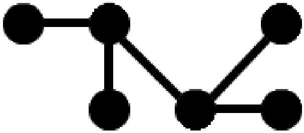

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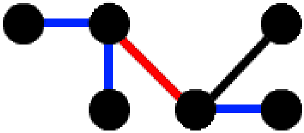

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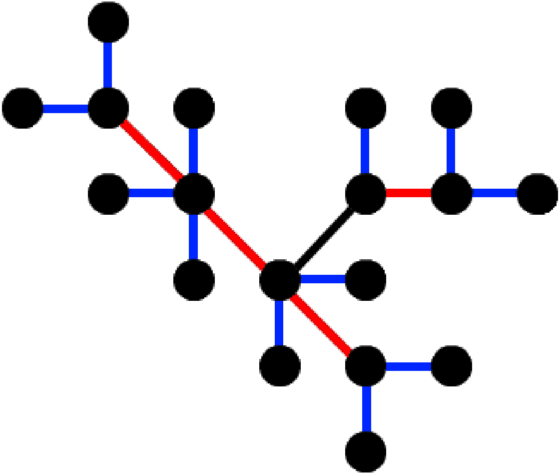

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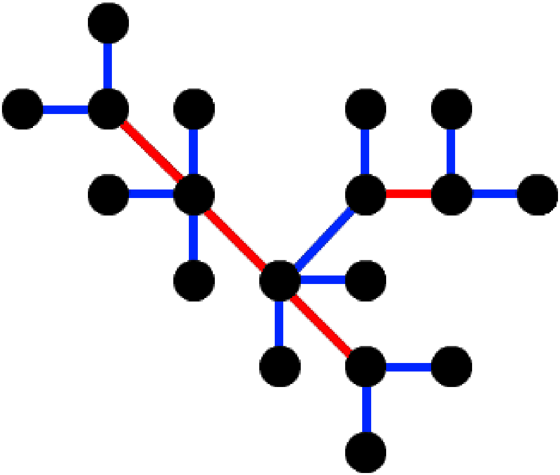

Richard A. Brualdi, Kathleen P. Kiernan, Seth A. Meyer, Michael W. Schroeder , *Patterns of Alternating Sign Matrices*, Department of Mathematics University of Wisconsin, 2011