Alternating Signed Bipartite Graphs

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The number of $n \times n$ *ASMs* is $\frac{1!4!7!...(3n-2)!}{n!(n+1)!(n+2)!...(2n-1)!}$.

ASBGs

Associated to each ASM is an *alternating signed bipartite graph*. This graph has a vertex for each row and column of the matrix. Vertex r_i is connected to vertex c_j by a positive edge (represented in blue) if there is a 1 in position (i, j) of the matrix, and by a negative edge (represented in red) if there is a -1 in position (i, j).

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$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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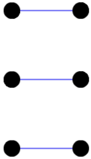
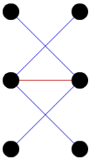


$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



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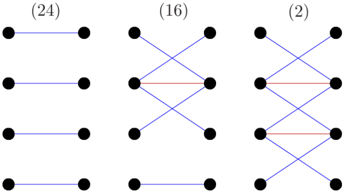

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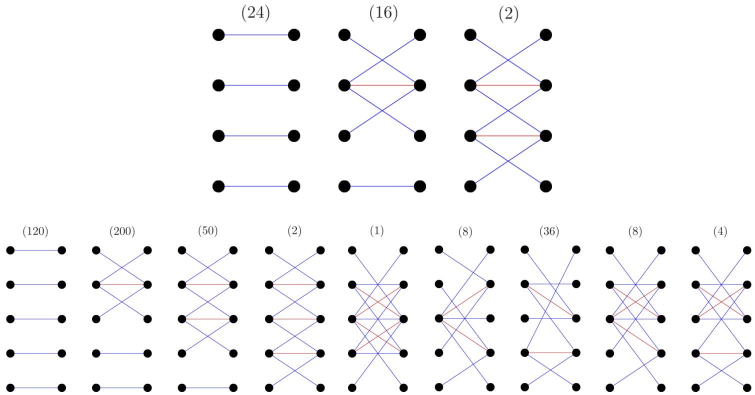
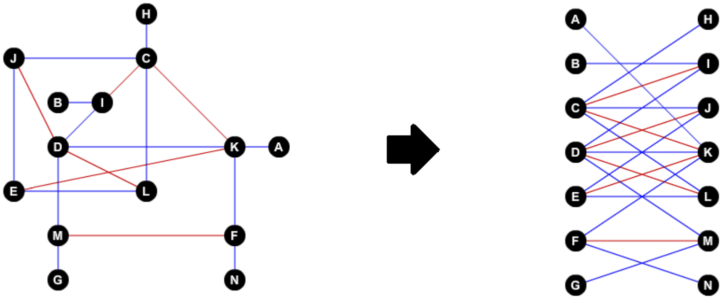


Identifying ASBGs



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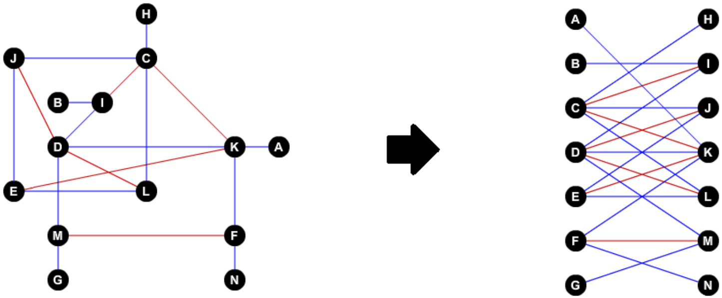

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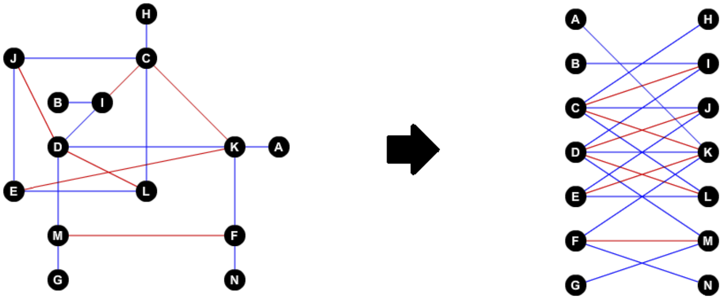

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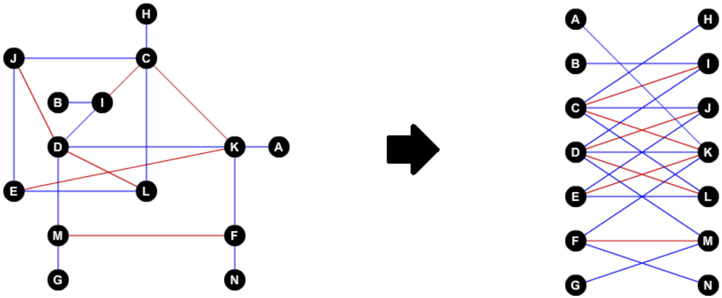

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- The graph must be bipartite
- The graph must be *balanced*
- $deg_b(v_i) = deg_r(v_i) + 1, \forall i = 1, 2, \ldots, 2n$

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$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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This is called the *elementary ASM expansion* of a matrix.

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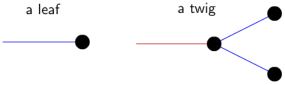

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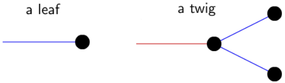
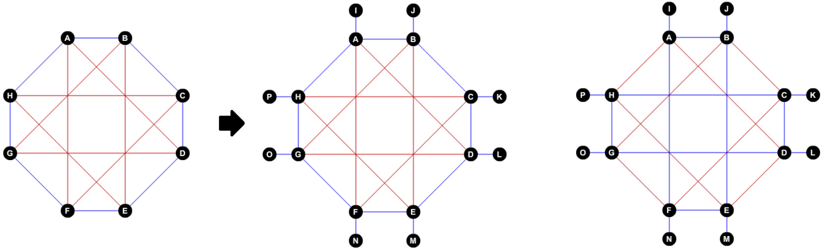


There are exceptions to this. For example, the following graph needs no further extension but is not an ASBG:



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When trying to determine if a graph is an ASBG, it is useful to define the *core* of a graph. The core of a graph is the subgraph that remains after removing leaves from the graph. There are restrictions on what the core of an ASBG can be:

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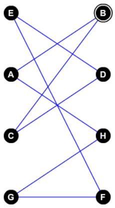

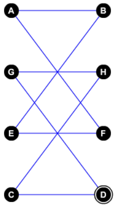

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