Computations Induced (Co)homology homomorphism

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Overview

- Resolution
- Chain and cochain complex
- homology
- cohomology
- Induce (Co)homology homomorphism

Resolution

Definition

Let G be a group and $\mathbb{Z}$ be the group of integers considered as a trivial $\mathbb{Z}G$-module. The map $\epsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$ from the integral group ring to $\mathbb{Z}$, given by $\Sigma m_g g \longmapsto \Sigma m_g$, is called the *augmentation*.

Definition

Let G be a group. A *free $\mathbb{Z}G$-resolution* of $\mathbb{Z}$ is an exact sequence of free $\mathbb{Z}G$-modules $R_*^G : \cdots \longrightarrow R_{n+1}^G \xrightarrow{\partial_{n+1}} R_n^G \xrightarrow{\partial_n} R_{n-1}^G \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R_1^G \xrightarrow{\partial_1} R_0^G \xrightarrow{\epsilon} R_{-1}^G = \mathbb{Z} \longrightarrow 0$
with each R_i^G a free $\mathbb{Z}G$-module for all $i \geq 0$.

Chain and cochain complex

- Given a $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$ and any $\mathbb{Z}G$-module A we can use the tensor product to construct an induced chain complex $R_*^G \otimes_{\mathbb{Z}G} A$ of abelian groups: $R_*^G \otimes_{\mathbb{Z}G} A : \cdots \longrightarrow R_{n+1}^G \otimes_{\mathbb{Z}G} A \xrightarrow{\partial_{n+1}} R_n^G \otimes_{\mathbb{Z}G} A \xrightarrow{\partial_n} R_{n-1}^G \otimes_{\mathbb{Z}G} A \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R_1^G \otimes_{\mathbb{Z}G} A \xrightarrow{\partial_1} R_0^G \otimes_{\mathbb{Z}G} A$

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- We can also construct an induced cochain complex $Hom_{\mathbb{Z}G}(R_*^G, A)$ of abelian groups:
  $Hom_{\mathbb{Z}G}(R_*^G, A) : \cdots \longleftarrow Hom_{\mathbb{Z}G}(R_{n+1}^G, A) \xleftarrow{\delta_n} Hom_{\mathbb{Z}G}(R_n^G, A) \xleftarrow{\delta_{n-1}} Hom_{\mathbb{Z}G}(R_{n-1}^G, A) \xleftarrow{\delta_{n-2}} \cdots \xleftarrow{\delta_1} Hom_{\mathbb{Z}G}(R_1^G, A) \xleftarrow{\delta_0} Hom_{\mathbb{Z}G}(R_0^G, A)$

Chain and cochain complex

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Definition

Let $C = (C_n, \partial_n)_{n\in\mathbb{Z}}$ be a *chain complex* of $\mathbb{Z}$-modules. For each $n \in \mathbb{Z}$, the *nth homology module* of C is defined to be the quotient module

$$H_n(C) = \frac{Ker\partial_n}{Im\partial_{n+1}}$$

Induce (Co)homology homomorphism

Definition

Given a chain complex and a group G, we can define the cochains C_n^* to be the respective groups of all homomorphisms from C_n to G: $C_n^* = Hom(C_n, G)$. We define the coboundary map $\delta_{n-1} : C_{n-1} \longrightarrow C_n$ dual to ∂_n. we can define the *nth* cohomology group as the quotient:

$$H^n(C_n^*) = \frac{ker\delta_n}{Im\delta_{n-1}}.$$

- Let C and D be chain complexes and let $f : C \longrightarrow D$ be a chain homomorphism. The commutativity of the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\partial} & C_{n-1} \\ {\scriptstyle f_n} \downarrow & & \downarrow f_{n-1} \\ D_n & \xrightarrow{\partial} & D_{n-1} \end{array}$$

Induce (Co)homology homomorphism

implies that f_n restrict to maps from $Z_n(C)$ into $Z_n(D)$ (where Z_n is cycle) and $B_n(C)$ into $B_n(D)$ (where B_n is boundary). Hence it induces a homomorphism, $f_* : H_n(C) \longrightarrow H_n(D)$ Then f_* is called the induced homomorphism on homology groups.

Induce (Co)homology homomorphism

- Let f' be the dual cochain homomorphism from C^* to D^*. As in the case of chain homomorphism, that we have a commutative diagram

$$\begin{array}{ccc} C^n & \xrightarrow{\delta} & C^{n+1} \\ f^n \downarrow & & \downarrow f^{n+1} \\ D^n & \xrightarrow{\delta} & D^{n+1} \end{array}$$

which implies f'^n restricts to maps from $Z^n(C)$ and $B^n(C)$ into $Z^n(D)$ and $B^n(D)$ (where Z^n is cocycle and B^n is coboundary), respectively, and so induces a homomorphism $f^* : H^n(C) \longrightarrow H^n(D)$ Then f^* is called the induced homomorphism on cohomology groups.

Induce (Co)homology homomorphism

Theorem

Let R_* be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Let A be a $\mathbb{Z}G$-module. There is an isomorphism

$R_n \otimes_{\mathbb{Z}G} A \cong \underbrace{A \times A \times \cdots \times A}_{r_n} = A^{r_n}$,

where $r_n = rank_{\mathbb{Z}G}(R_n)$.

Example

Consider the symmetric group $G = S_4$ and the identity map $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$. The following commands then compute the homology homomorphism

$$H_3(S_4, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{H_3(\phi)} H_3(S_4, \mathbb{Z}/4\mathbb{Z})$$

and determine that the kernel of this homomorphism has Size 1.

Example

GAP session

```
gap > G := SymmetricGroup(4);;
gap > a := AbelianGroup([4]);;
gap > b := a;;
gap > ahomb := GroupHomomorphismByFunction(a, b, x− >
x);;
gap > A := TrivialGModuleAsGOuterGroup(G, a);;
gap > B := TrivialGModuleAsGOuterGroup(G, b);;
gap > phi := GOuterGroupHomomorphism();;
gap > phi!.Source := A;;
gap > phi!.Target := B;;
gap > phi!.Mapping := ahomb;;
gap > Hphi := HomologyHomomorphism(phi, 3);;
gap > Size(KernelOfGOuterGroupHomomorphism(Hphi));
1
```

Summary

- We devise and implement an algorithm for computing the induced homology homomorphism $H_n(\phi) : H_n(G, A) \longrightarrow H_n(G, B)$.
- We devise and implement an algorithm for computing induced cohomology homomorphism $H^n(\phi) : H^n(G, A) \longrightarrow H^n(G, B)$.

Bibliography I

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1994.

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