Dealing with Discontinuity Boundaries

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Linearisation of Smooth Systems

In a smooth dynamical system the characteristics of a given reference trajectory can be determined by examining the linearised system about the reference trajectory. Suppose the system

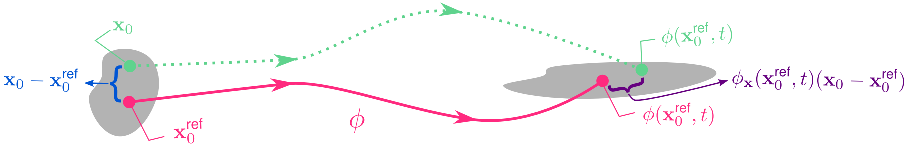
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = \mathbf{x}_0, \quad \text{has the unique solution} \quad \mathbf{x}(t) = \phi(\mathbf{x}_0, t). \tag{1}$$

Then for $\mathbf{x}_0$ in a small neighbourhood of $\mathbf{x}_0^{\text{ref}}$

$$\phi(\mathbf{x}_0, t) - \phi(\mathbf{x}_0^{\text{ref}}, t) = \phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)(\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}) + \mathcal{O}(\|\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}\|), \tag{2}$$

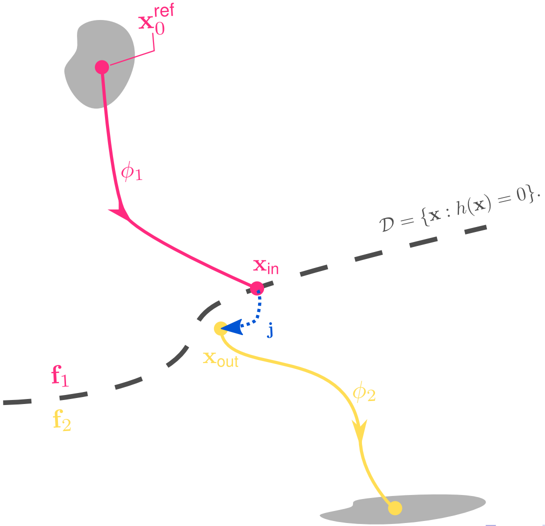
where the Jacobian $\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)$ is the solution to the IVP

$$\dot{\mathbf{\Phi}} = \mathbf{f}_{\mathbf{x}}(\phi(\mathbf{x}_0^{\text{ref}}, t))\mathbf{\Phi}, \quad \mathbf{\Phi}(0) = Id. \tag{3}$$



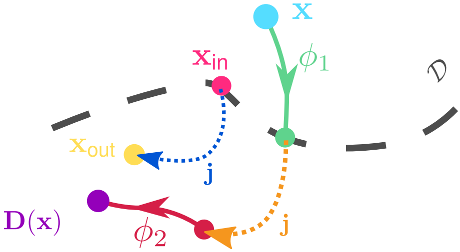
Nonsmooth Systems

This form of analysis cannot be used in nonsmooth systems as $\mathbf{f}$ is not everywhere differentiable, or $\phi(\mathbf{x}_0^{\text{ref}}, t)$ is not continuous.



Zero-Time Discontinuity Mapping

To account for this we derive the zero-time discontinuity mapping (ZDM).



Let $t(\mathbf{x})$ be the time of flight from $\mathbf{x}$ to the boundary and $t(\mathbf{x}_0^{\text{ref}}) = t_{\text{ref}}$. Using this we can write

$$\phi(\mathbf{x}_0, T) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), T - t_{\text{ref}}), \tag{4}$$

where the ZDM

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x})) \tag{5}$$

takes a point in a neighbourhood of $\mathbf{x}_{\text{in}}$ and maps it to a point in a neighbourhood of $\mathbf{x}_{\text{out}}$.

The Saltation Matrix

The Jacobian derivative of $\mathbf{D}$ evaluated at $\mathbf{x}_{\text{in}}$ is given by

$$\begin{aligned}\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) &= \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) + (\mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) - \mathbf{f}_{\text{out}})t_{\mathbf{x}}(\mathbf{x}_{\text{in}}) \\ &= \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) + \frac{(\mathbf{f}_{\text{out}} - \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}})h_{\mathbf{x}}(\mathbf{x}_{\text{in}})}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}},\end{aligned} \tag{6}$$

where $\mathbf{f}_{\text{in}} = \mathbf{f}_1(\mathbf{x}_{\text{in}})$ and $\mathbf{f}_{\text{out}} = \mathbf{f}_2(\mathbf{x}_{\text{out}})$. In the case where h is explicitly time-dependent this becomes

$$\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) = \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) + \frac{(\mathbf{f}_{\text{out}} - \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}})h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})}{h_t(\mathbf{x}_{\text{in}}, t_{\text{ref}}) + h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}}}. \tag{7}$$

In both cases we have that

$$\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, T) = \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, T - t_{\text{ref}})\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\phi_{1,\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}}). \tag{8}$$

A PWL Example

Consider the piecewise linear system in the plane given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{f_1}(\mathbf{x}) = \mathbf{A}_L\mathbf{x} + \mathbf{u}_L \text{ if } \mathbf{x} \in S^-, \\ \mathbf{f_2}(\mathbf{x}) = \mathbf{A}_R\mathbf{x} + \mathbf{u}_R \text{ if } \mathbf{x} \in S^+, \end{cases} \tag{9}$$

where the two regions

$$S^- = \{\mathbf{x} = (x, y) \colon x < 0\} \quad \text{and} \quad S^+ = \{\mathbf{x} = (x, y) \colon x \geq 0\} \tag{10}$$

are separated by the discontinuity boundary

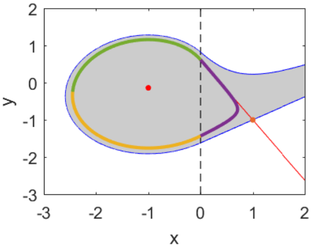
$$\mathcal{D} = \{\mathbf{x} = (x, y) \colon h(\mathbf{x}) = x = 0\}. \tag{11}$$

Take

$$\mathbf{A}_L = \begin{pmatrix} 2\gamma & -1 \\ \gamma^2 + 1 & 0 \end{pmatrix}, \ \mathbf{u}_L = \begin{pmatrix} 0 \\ \gamma^2 + 1 \end{pmatrix}, \ \mathbf{A}_R = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix}, \ \mathbf{u}_R = \begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{12}$$

It has been shown by Ponce et al [PRV13] that, in this system when $a > 0$ and $T < 0$, a stable limit cycle bifurcates as γ increases through 0 in a *focus-center-limit cycle bifurcation*. This limit cycle exists provided γ is sufficiently small.

A PWL Example



The saltation matrices are given by

$$\mathbf{D}_{i,\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ \frac{\gamma^2+1-a}{(-1)^{i+1}y_i} & 1 \end{pmatrix}, \text{ for } i = 1, 2, \quad (13)$$

and the Jacobians of the flows are given by

$$\phi_{L,\mathbf{x}}(t) = e^{\gamma t} \begin{pmatrix} \cos(t) + \gamma \sin(t) & -\sin(t) \\ (1+\gamma^2)\sin(t) & \cos(t) - \gamma \sin(t) \end{pmatrix} \quad (14)$$

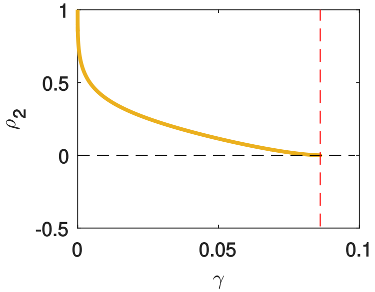
and

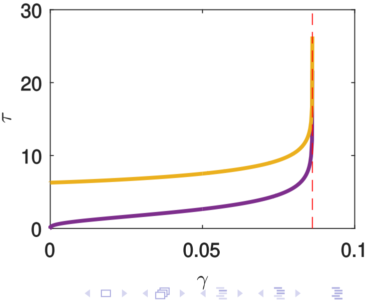
$$\phi_{R,\mathbf{x}}(t) = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ D(e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} \end{pmatrix} \quad (15)$$

where $\Delta = T^2 - 4D$ and $\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{\Delta})$.

A PWL Example

The eigenvalues of the Jacobian of the periodic orbit are known as the *characteristic multipliers* or *Floquet* multipliers, ρ_i of the periodic orbit. The multiplier associated with perturbations along the periodic solution is always unity, we will label this multiplier ρ_1. The value of the remaining multipliers, or multiplier in this case as the system under consideration is 2-dimensional determines the local stability of the periodic orbit. In particular if the remaining multipliers have magnitude less than unity the periodic orbit under consideration is stable.





Introducing Noise to the Boundary

We are interested in deriving the saltation matrix of the system in the case where the vector fields on either side of the discontinuity boundary are once again entirely deterministic and locally but the boundary varies randomly in time.

- Derive an appropriate stochastic process.
  - Mean-reverting with mean 0.
  - At least once differentiable.
  - Does not depend on $\mathbf{x}$.
- Derive a stochastic saltation matrix.
  - Contain the entire effect of both the discontinuity and the randomness.
  - Composable with the deterministic Jacobians of the two flows to give overall Jacobian.
- Use this to analyze the effect of noise on the behaviour of the system.
  - Stability of periodic orbits.
  - Effect on bifurcations.
  - Inducing new behaviours.

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Enrique Ponce, Javier Ros, and Elísabet Vela, *The focus-center-limit cycle bifurcation in discontinuous planar piecewise linear systems without sliding*, Progress and Challenges in Dynamical Systems, Springer, 2013, pp. 335–349.