Noise, Hysteresis, Time Delay and the Resolution of a Two-Fold Singularity

Eoghan Staunton, Petri T. Piiroinen

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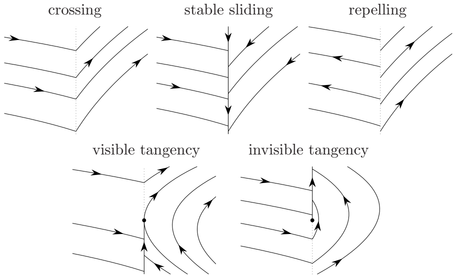
Introduction

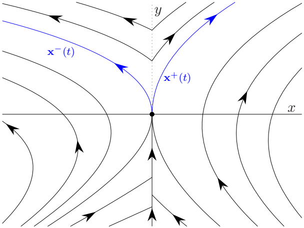
Systems of PWS ODEs are utilised in many areas to model phenomena involving switching events. Except in special cases, the dynamical behaviour local to a point on a discontinuity surface conforms to one of three scenarios: crossing, stable sliding, or repelling.

Points on discontinuity surfaces at which the vector field is tangent to the surface on one side usually represent boundaries at which the nature of the discontinuity surface changes between one of the three generic scenarios.



The Two-Fold Singularity

Points on discontinuity surfaces at which the vector field is tangent to the surface on both sides are known as two-folds. Forward evolution from two-folds is generally ambiguous.
We will look at some physically motivated perturbations of a discontinuous ODE system with the aim of resolving the ambiguity of forward evolution at twofolds.



- **Hysteresis** - Replace the discontinuity surface with a hysteretic band of width 2ε.
- **Time Delay** - Forward evolution switches at a time ε after trajectories intersect the discontinuity surface.
- **Noise** - Additive white Gaussian noise of amplitude ε is added to the system.

The Two-Fold Singularity

By choosing coordinates, (x, y), such that the discontinuity surface coincides with the y-axis, a description of the local dynamics of the system may be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} f^{(L)}(x,y) \\ g^{(L)}(x,y) \end{bmatrix}, & x < 0; \\ \begin{bmatrix} f^{(R)}(x,y) \\ g^{(R)}(x,y) \end{bmatrix}, & x > 0. \end{cases} \tag{1}$$

In order for the system to exhibit a visible two-fold at the origin like the one shown on the previous slide we must have

$$\begin{aligned} f^{(L)}(0,0) = 0, &\quad f^{(R)}(0,0) = 0, \\ \frac{\partial f^{(L)}}{\partial y}(0,0) < 0, &\quad \frac{\partial f^{(R)}}{\partial y}(0,0) > 0, \\ g^{(L)}(0,0) > 0, &\quad g^{(R)}(0,0) > 0. \end{aligned} \tag{2}$$

The Two-Fold Singularity

Rescaling and expanding the two halves of the vector field as Taylor series centred about the origin we can rewrite the system as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} -\mathcal{A}y + \mathcal{O}(|x|) + \mathcal{O}(2) \\ \mathcal{B} + \mathcal{O}(1) \end{bmatrix}, & x < 0; \\ \begin{bmatrix} y + \mathcal{O}(|x|) + \mathcal{O}(2) \\ 1 + \mathcal{O}(1) \end{bmatrix}, & x > 0. \end{cases} \tag{3}$$

where

$$\mathcal{A} = -\frac{\frac{\partial f^{(L)}}{\partial y}(0,0)}{\frac{\partial f^{(R)}}{\partial y}(0,0)}, \quad \mathcal{B} = \frac{g^{(L)}(0,0)}{g^{(R)}(0,0)}. \tag{4}$$

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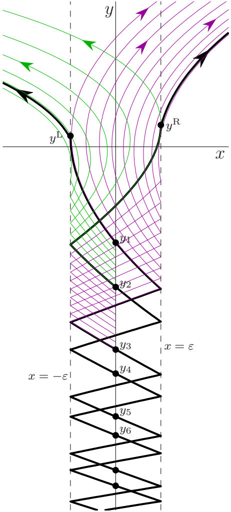
Hysteresis



Stable sliding motion approaching the two-fold is replaced by rapid switching motion. For any $\varepsilon > 0$, forward evolution is uniquely determined in a neighbourhood of the two-fold.

Forward orbits pass close to the two-fold and then are directed away from the two-fold along a path close to either $x^+(t)$ or $x^-(t)$.

We are interested in the fraction of these orbits that go right, i.e. follow a path close to $x^+(t)$, in the limit $\varepsilon \to 0$, denoted $\mathcal{Q}_{hy}$.

Hysteresis

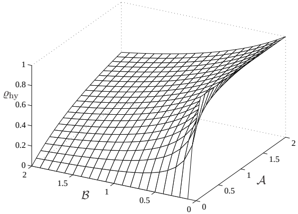
There are two critical trajectories that separate orbits by whether they eventually head right or eventually head left. These trajectories have a tangential intersection with one of the hysteretic switching manifolds.

For any interval of y-values I let q_I be the fraction of trajectories that head right. Then

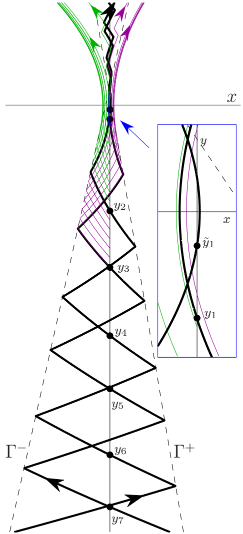
$$q_{[y_k, y_{k-2}]} = \frac{y_{k-1} - y_k}{y_{k-2} - y_k}, \quad k = 3, 5, 7, \dots \tag{5}$$

and we are interested in

$$\mathcal{Q}_{hy} = \lim_{\varepsilon \to 0} q_{[y_{min}, y_{max}]} = \lim_{k \to \infty} \lim_{\varepsilon \to 0} q_{[y_k, y_{k-2}]} = \frac{\mathcal{A}}{\mathcal{A} + \mathcal{B}}. \tag{6}$$



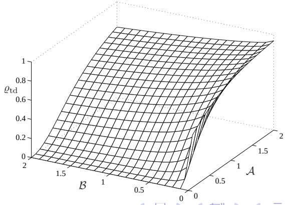
Time Delay



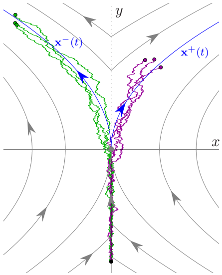
Most orbits passing close to the two-fold eventually stop switching and either head right or head left. However, there are two critical orbits that switch across the positive y-axis indefinitely.
Again we have

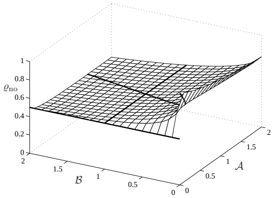
$$q_{[y_k,y_{k-2}]} = \frac{y_{k-1} - y_k}{y_{k-2} - y_k}, \quad k = 3, 5, 7, \dots \tag{7}$$

and we are interested in the fraction of these orbits that go right in the limit $\varepsilon \to 0$, denoted $\mathcal{Q}_{td}$.



Noise





Once again our interest is in forward evolution from points on the negative y-axis. For small $\varepsilon > 0$, upon passing near the two-fold, sample paths follow close to either $x^+(t)$ or $x^-(t)$ with high probability. We restrict our attention to a sufficiently ε-independent rectangle about the two-fold. For any initial point, $(x(0), y(0)) = (x0, y0)$, inside the rectangle, we let $Q_\varepsilon(x0, y0)$ denote the probability that first passage by to the boundary of the rectangle occurs at a point with $x > 0$.

$$\mathcal{Q}_{no} = \lim_{\varepsilon \to 0} Q_\varepsilon(0, y0), \tag{8}$$

i.e., the probability that forward evolution from the negative y-axis heads right after passing by the origin in the zero-noise limit.

References

David J. W. Simpson, *On resolving singularities of piecewise-smooth discontinuous vector fields via small perturbations*, Discrete and Continuous Dynamical Systems **34** (2014), no. 9, 3803–3830.