Noise and Multistability

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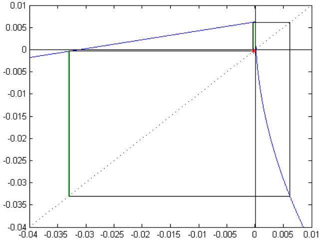
Nordmark's Square Root Map

Many impacting systems are described by a one-dimensional map known as the Nordmark square root map. The map is derived as an approximation for solutions of piecewise smooth differential equations near certain types of grazing bifurcation.



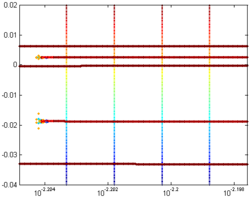
A grazing bifurcation is an effect of the fact that an impact with low velocity is sensitive to small changes in the initial conditions, with sensitivity inversely proportional to impact velocity.

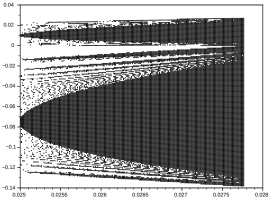
$$x_{n+1} = S(x_n) = \begin{cases} \mu + b x_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases} \tag{1}$$

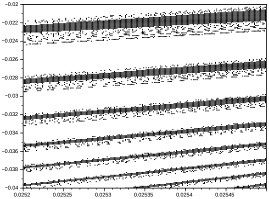
Multistability In the Square Root Map

If $0 < b < \frac{1}{4}$ there are values of $\mu > 0$ for which a stable periodic orbit with code $(RL^n)^\infty$ exists for each $n = 1, 2, \ldots$, and also such that there are two stable periodic orbits, one with code $(RL^n)^\infty$ and the other with code $(RL^{n+1})^\infty$.

For b in this range these are the only possible attractors except at bifurcation points.







Types of Noise

Simpson and Kuske make a careful analysis of how noise in impacting systems manifests in the map. They conclude that there are three different models:

1. if the noise is in the impact itself then the constant a is replaced (in the limit of small noise) by $a(1 + \frac{1}{2}\xi_k)$, i.e. the effect of this is parametric noise;
2. if there is noise in the position of the impacting surface then the switch at $x = 0$ is replaced by a sum $x_k + A\xi_k = 0$ and this variable is used in the square root for the impact too.
3. In both of the above they consider coloured noise, the third case they consider is the small correlation time limit i.e. white noise on the impacting dynamics.

In an earlier paper Simpson, Hogan and Kuske consider additive Gaussian noise of amplitude, Δ, and show that this particular noise formulation arises in a general setting.

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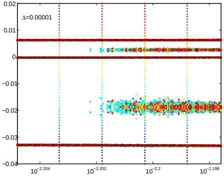
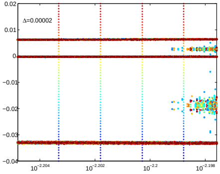
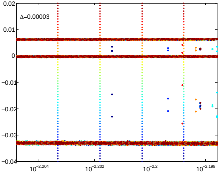
The Effect of Noise

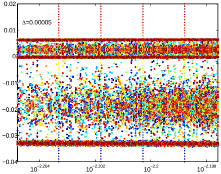
My work thus far has focused on phase space sensitivity for period two and three coexistence, investigating a shift of the proportion of points going to one behaviour or the other, for both parametric and additive noise.

The results have not been entirely as we had expected. The relationship between the proportion of points going to each of the coexisting attractors is not monotonic.









The Effect of Noise

This is understandable as we can think of any noise as an error in the bifurcation parameter μ, independent of the rest of the dynamics in the case of additive white noise and depending on $\frac{a}{2}\sqrt{x}$ in the case of parametric white noise.

$$x_{n+1} = S_a(x_n) = \begin{cases} (\mu + \xi_n) + bx_n & \text{if } x_n < 0 \\ (\mu + \xi_n) - a\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases} \tag{2}$$

$$x_{n+1} = S_p(x_n) = \begin{cases} \mu + bx_n & \text{if } x_n < 0 \\ (\mu - \frac{a}{2}\sqrt{x_n}\xi_n) - a\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases} \tag{3}$$

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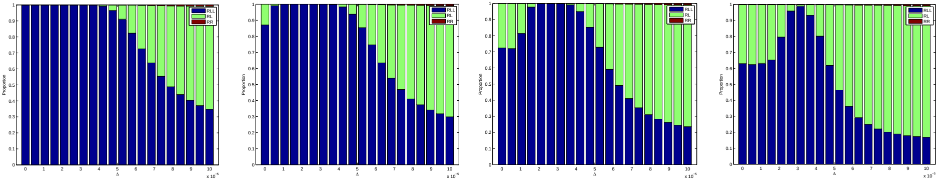
Proportions

The bifurcation diagrams we have seen lead us to believe that - for fixed μ close to μ_2^s - with increasing noise amplitude we first see a decrease in the probability of being in RL behaviour to some minimum followed by an increase in this probability as μ increases further.

Looking at both the proportion of points in RL behaviour at a certain point in time, and the proportion of time spent in RL behaviour over a long period, we have confirmed this for both additive and parametric noise.



Mechanics

Perhaps the most interesting phenomenon that we have observed is the potential for repeated intervals of persistent RL dynamics in a noisy system with $\mu < \mu_2^s$.
In the case of both additive and parametric noise, we have observed that the noise-induced transition between RLL and RL behaviour takes the following symbolic form

$$RLLRLL \ldots RLL\underline{RLLRLRRL}RL \ldots RL. \tag{4}$$

The most significant feature of the transition is the repeated R, corresponding to repeated low velocity impacts.
These repeated low velocity impacts allow the dynamics to be pushed in to the region of phase space with slow dynamics, in the vicinity of the unstable $(RL)^\infty$ orbit of the deterministic system.

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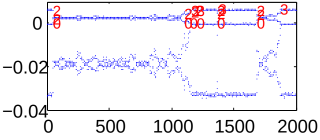
Mechanics

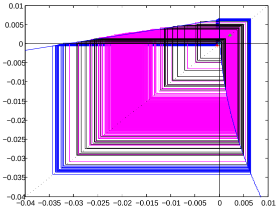
Let us label the iterates represented by the underlined portion of the symbolic sequence (4) $x_1, x_2, \ldots x_8$ respectively. For the transition to take place in this manner the following conditions must be satisfied.

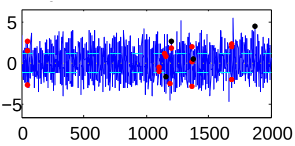
$$(\mu + \xi_5) + bx_5 > 0 \quad (5)$$

$$(\mu + \xi_6) - a\sqrt{x_6} > 0 \quad (6)$$

$$(\mu + \xi_7) - a\sqrt{x_7} < 0. \quad (7)$$







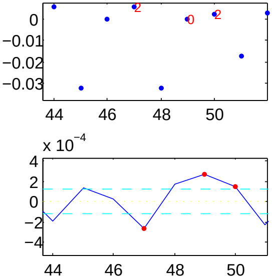
Observations

In the case of additive noise, we have observed some reoccurring characteristics of the noisy signal associated with the transition from RLL to RL behaviour. In general we find that

$$\xi_4 < \xi_5 < \xi_6 > \xi_7 \quad (8)$$

$$\xi_4 < 0 \quad (9)$$

$$\xi_6 > 0. \quad (10)$$



The case of parametric noise is more complicated due to its nonlinearity. Although the transition occurs through the same symbolic sequence we have not yet been able to pinpoint any particular features of the noise that leads to such dynamics.

Future Work

- We plan to look at mapping these regions to understand the space we are in when these transition dynamics occur.
- By using this method we may also get a handle on why the sequence $RLL\ldots RLLRLRL\ldots RLRL$ does not appear to occur (is this a small region?).
- We would like to investigate what conditions are required on the noise to maintain RL behaviour for a significant number of iterates after the transition.
- How does this problem scale in relation to noise amplitude as we look at regions of RL^n and RL^{n+1} coexistence for increasing n.

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References

A.B. Nordmark, *Universal limit mapping in grazing bifurcations*, Phys. Rev. E **55** (1997), 266–270.

D.J.W. Simpson, S.J. Hogan, and R. Kuske, *Stochastic regular grazing bifurcations*, SIADS **12** (2013), 533–559.

D.J.W. Simpson and R. Kuske, *The influence of localised randomness on regular grazing bifurcations with applications to impact dynamics*, Journal of Vibration and Control **12** (2016), 1077546316642054.

E.J. Staunton, P.T. Piiroinen, and P. Glendinning, *Noise and multistability in impacting systems*, In Preparation (2016).