The Importance of Noise in Dynamical Systems

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Introduction

Historically mathematicians have made widespread use of smooth, deterministic mathematical models to describe real-world phenomena. These models present a simplified view of the world where

1. The future of any system is completely determined by its present state.
2. The evolution of systems is always smooth and exhibits no interruptions such as impacts, switches, slides or jumps.

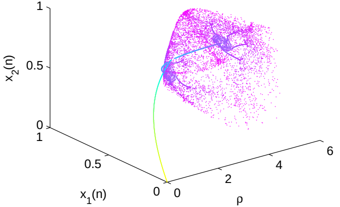

Figure: Bifurcation diagram for a smooth dynamical system on a network with two nodes

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The Importance of Noise

- All real-world systems evolve in the presence of noisy driving forces.
- Often thought that noise has only a blurring effect on the evolution of dynamical systems.
  - This can be the case, especially in the case of
    1. "measurement" noise
    2. linear systems.
- In nonlinear systems with dynamical noise the deterministic dynamics can be drastically modified.

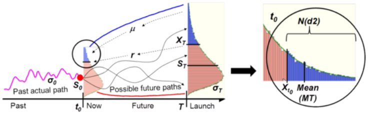

Figure: Models for option valuation.

Noise and Bifurcations

- Noise has its greatest influence in the vicinity of bifurcation points, the hallmark of nonlinear behaviour. Noise
    1. can make the determination of bifurcation points very difficult, even for the simplest bifurcations.
    2. can shift bifurcation points.
    3. can induce behaviors that have no deterministic counterpart, through what are known as noise-induced transitions.
    4. can produce time series that are easily mistakable for deterministic chaos near bifurcation points.

Noise and the Logistic Map

Investigating the statistical dynamics of the response of the logistic map to the inclusion of noise in a range of the control parameter r where the first bifurcation of the unperturbed system is located yields some interesting results.
We couple the stochastic forces ξ_n with the variable x additively

$$x_{n+1} = rx_n(1 - x_n) + \Delta\xi_n \tag{1}$$

where the stochastic forces have vanishing mean and unit covariance and Δ measures the intensity of the noise.

Noise and the Logistic Map

Expanding about $(x, \Delta) = (x^*, 0)$ where x^* is the stable fixed point and writing

$$x_n = x_n^{(0)} + \Delta x_n^{(1)} + \Delta^2 x_n^{(2)} + \dots, \tag{2}$$

we can decompose the equation of motion (1) up to second order in Δ into the system

$$x_{n+1}^{(0)} = x^*, \tag{3}$$

$$x_{n+1}^{(1)} = q x_n^{(1)} + \xi_n, \tag{4}$$

$$x_{n+1}^{(2)} = q x_n^{(2)} - r x_n^{(1)}, \tag{5}$$

with

$$q = r(1 - 2x^*). \tag{6}$$

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Noise and the Logistic Map

Now for noise with a Brownian two-point correlation spectrum we find the stationary mean

$$\langle x(\Delta)\rangle = x^* - \Delta^2 \frac{r}{(1-q)^2(1+q)}\left(1 + 2\frac{qe^{-\gamma}}{1-qe^{-\gamma}}\right) + \mathcal{O}(\Delta^4) \qquad (7)$$

This is negative for $0 < r < 1$ with its absolute size increasing as r approaches 1, there our small Δ expansions lose their validity. However (7) agrees well with numerical simulations provided x_n does not diverge which is the case whenever the distance $|r-1|$ is large enough to prevent the presence of noise moving the trajectory into the basin of attraction of the deterministic fixed point $x = -\infty$.

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Noise and the Logistic Map

Agreement of the solid curves in the numerical simulation with the small-Δ, result for white noise, $\gamma = \infty$, is perfect.

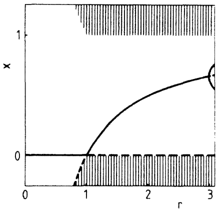

Figure: Bifurcation diagram and basins of attraction.

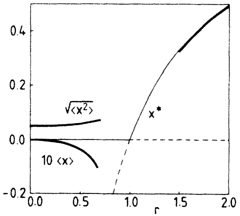

Figure: Additive noise ($\Delta = 0.05$) destroys the transcritical bifurcation at r= 1 in the logistic map.

A Physiological Example - The Pupil Light Reflex

The pupil light reflex is modelled deterministically by

$$\frac{dA}{dt} = -\alpha A + \frac{c}{1 + \left[\frac{A(t-\tau)}{\theta}\right]^n} + k \tag{8}$$

where A(t) is the pupil area, and the second term on the right is a sigmoidal negative feedback function of the area at a time τ msec in the past.

The deterministic system predicts a single stable fixed point which becomes unstable, giving rise to a stable limit cycle if τ is increased beyond some critical delay (or n beyond some critical n_0) (super-critical Hopf bifurcation). This behaviour is not observed in experiments instead oscillations are observed even for the lowest values of n and τ.

A Physiological Example - The Pupil Light Reflex

This is due to inherent noise present in the reflex. the pupil has a well documented source of fluctuations known as pupillary hippus.
Including noise we see a model that matches observations extremely well.
We also find that the inclusion of noise postpones the Hopf bifurcation, inducing stability in the statistical fixed point.

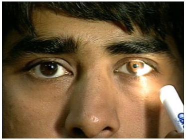

Figure: Pupil Light Reflex.

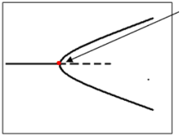

Figure: Super-critical Hopf Bifurcation.

Important Considerations when Including Noise

- How we include noise
  1. additive
  2. multiplicative
  3. other choices for nonsmooth systems
- The type of noise we include
  1. white noise
  2. real, coloured noise
- The intensity of the noise

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