Numerical Analysis of Singularly Perturbed Differential Equations

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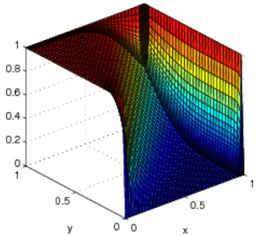
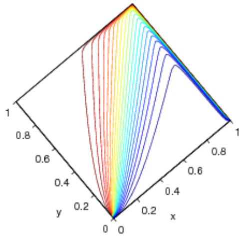
November 18, 2016





When is a perturbation singular?

A regular perturbation

Consider the following example, taken from [O'Malley, 1997]:

$$x^2 + \varepsilon x - 1 = 0. \tag{1}$$

Here ε is the *perturbation parameter*. It is real and positive. In cases of interest it is small.

The solutions to (1) are

$$x = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}. \tag{2}$$

If we let $\varepsilon \to 0$ in (1), the resulting problem has two solutions: $x = \pm 1$.

If we let $\varepsilon \to 0$ in (2), we again get $x = \pm 1$.

This is a regular perturbation

A singular perturbation

Now consider a similar problem, but with the perturbation parameter multiplying the second-order term:

$$\varepsilon x^2 + x - 1 = 0. \tag{3}$$

The solutions to this problem are

$$x = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}. \tag{4}$$

If we set $\varepsilon = 0$ in (3), the resulting problem has a single solution: $x = 1$.

But if we let $\varepsilon \to 0$ in (4), the solutions tend to 1 and $-\infty$.

This is a singular perturbation

(*A similar explanation is given by Peter D. Miller (Michigan) in "Perturbation theory and asymptotics", §IV.5 of The Princeton Companion to Applied Mathematics.*)

Compare the following two differential equations:

$$-u''(x) + \varepsilon u(x) = f(x) \quad \text{on } (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{5}$$

and

$$-\varepsilon u''(x) + u(x) = f(x) \quad \text{on } (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{6}$$

If we set $\varepsilon = 0$ in (5), nothing remarkable happens: we still have a well-posed ODE.

But if we set $\varepsilon = 0$ in (6), the problem is not well-posed, since, unless $f(0) = f(1) = 0$, we cannot satisfy $u = f$ and the boundary conditions.

Question: What happens to (6) as $\varepsilon \to 0$?
Answer: Solutions develop "layers".

Singularly perturbed problems [Roos et al., 2008, p2]

[**Singularly perturbed problems**] are differential equations (ordinary or partial) that depend on a small positive parameter, ε, and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be singularly perturbed, where we regard ε as a **perturbation parameter**.

Our first example is a simple *reaction-diffusion* equation.

A singularly perturbed differential equation

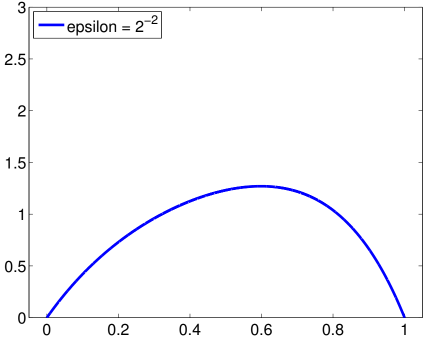
$$-\varepsilon^2 u''(x) + u(x) = e^x \quad \text{on } (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{7}$$

The solutions to this equation look like

$$\underbrace{c_0 e^{-x/\varepsilon}}_{\text{left layer}} + \underbrace{c_1 e^{-(1-x)/\varepsilon}}_{\text{right layer}} + \underbrace{(e^x)/(1-\varepsilon^2)}_{\text{regular part}}$$

The first two terms are "layer terms", which decay rapidly away from the boundaries.

The third term is close to the solution of the "reduced" problem, obtained by setting $\varepsilon = 0$, and neglecting the boundary conditions.



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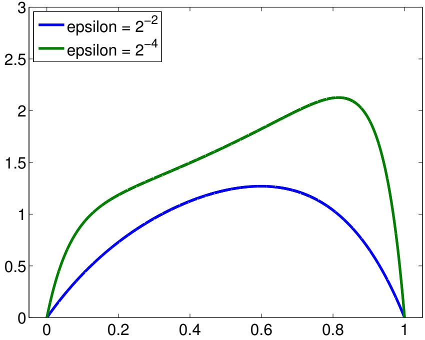

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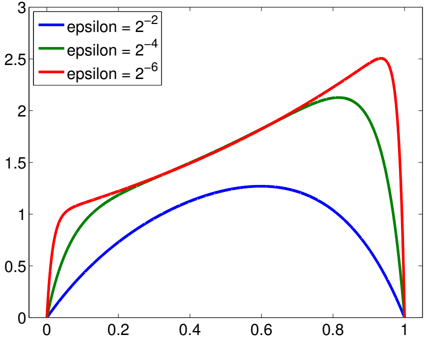

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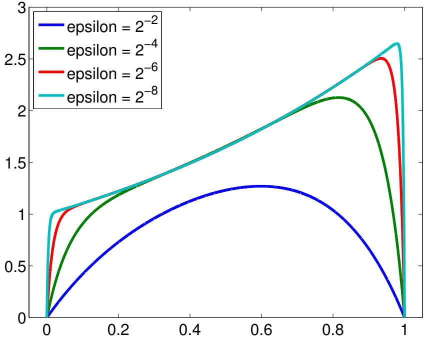


Types of singularly perturbed problem

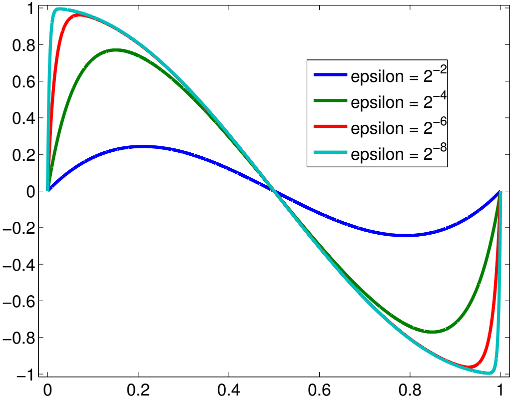
Here follows an incomplete list of SPDEs, with graphs of their solutions, and some notes about what makes them interesting.

This will include

1. reaction-diffusion ODEs;
2. convection-diffusion problems;
3. coupled systems (if I have time);
4. all the above, but in two dimensions: my next talk!

Example (A reaction-diffusion equation)

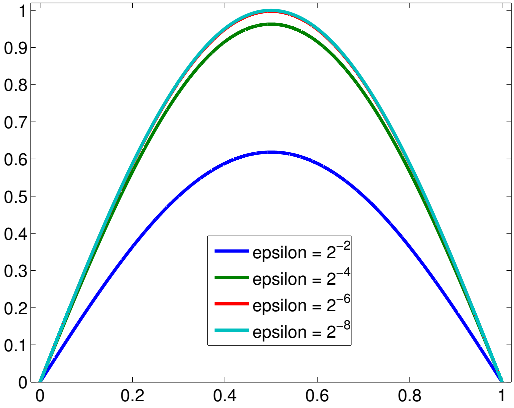
$$-\varepsilon^2 u''(x) + u(x) = \cos(\pi x) \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$



- Solution features layers of width $\mathcal{O}(\varepsilon)$ near $x = 0$ and $x = 1$.
- Away from layers $u \approx \cos(\pi x)$.

Example (Another reaction-diffusion equation)

$$-\varepsilon^2 u''(x) + u(x) = \sin(\pi x) \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$



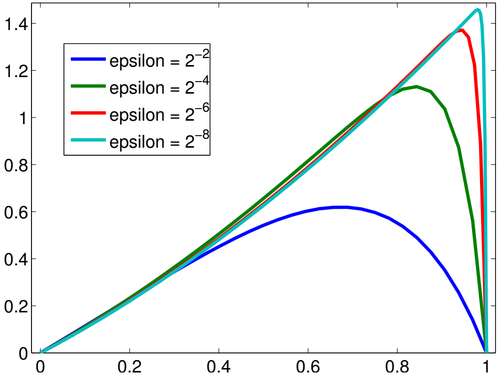
- The solution to the reduced equation satisfies the boundary conditions.
- The solution does not feature layers. In fact, $u = \sin(\pi x)/(\pi^2 \varepsilon^2 + 1)$.

Example (A convection-diffusion equation)

$$-\varepsilon u''(x) + u'(x) = x + 1 \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$

The solution to this problem is

$$\frac{\varepsilon + 3/2}{1 - e^{-1/\varepsilon}} \left(e^{-1/\varepsilon} - e^{-(1-x)/\varepsilon} \right) + \varepsilon x + x^2/2 + x;$$



- Notice that the diffusion coefficient is ε, and not ε^2.
- The solution possesses a single layer, near $x = 1$.
- Elsewhere, the solution resembles that of

$$u' = x + 1.$$

- Computing stable solutions can be a challenge for this problem.

The study of these simple-looking ODEs can rapidly become rather complex when extended to coupled systems. Our simplest example has just two equations.

Example (A coupled system of reaction-diffusion equations)

$$-\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}^2 \mathbf{u}'' + B(x)\mathbf{u} = \mathbf{f} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

There are many variants possible for this problem, including

1. Systems of $\ell > 1$ equations;
2. Systems of convection-diffusion equations;
3. Strongly coupled systems;

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In spite of its simplicity, there is much that can be learned from this problem, which itself is often reduced to three sub-classes:

(a) $\varepsilon_1 = \varepsilon_2 \ll 1$

(b) $\varepsilon_1 \ll \varepsilon_2 = 1$

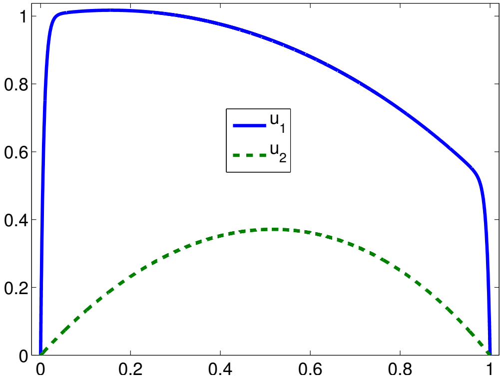
(c) $\varepsilon_1 \ll \varepsilon_2 \ll 1$

Case (a) is the least interesting. Under reasonable assumptions on B, most techniques (numerical and mathematical) for uncoupled problems extend directly to this case.

Outline

Example (Case (b): $\varepsilon_1 \ll \varepsilon_2 = 1$)

$$-\begin{pmatrix} 10^{-2} & 0 \\ 0 & 1 \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$



- The component u_1 features (strong) layers, of width $\mathcal{O}(\varepsilon)$.
- u_2 features "weak" layers: u_2' and u_2'' are bounded independent of ε, but u_2''' is not.

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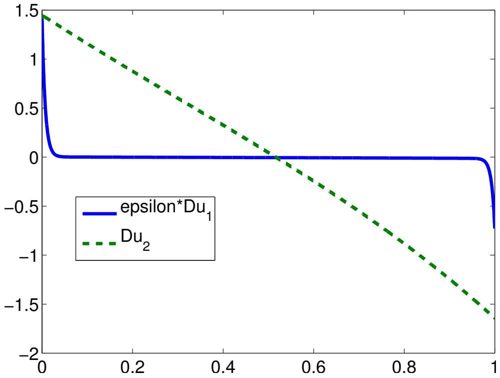

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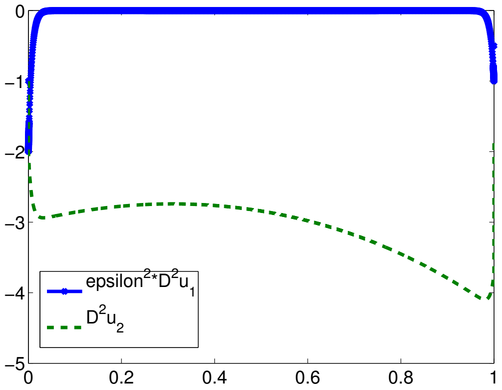

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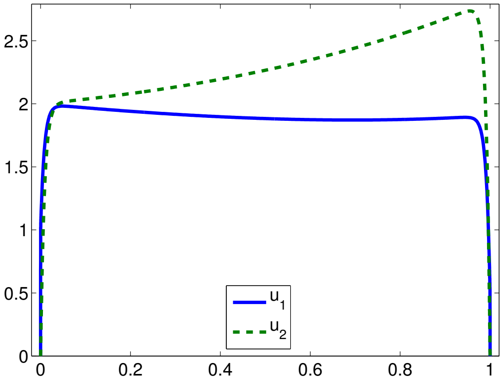
Outline

This is the most interesting case, since solutions possess multiple, interacting layers.

Example (Case (c): $\varepsilon_1 \ll \varepsilon_2 \ll 1$)

$$-\begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^{-2} \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \mathbf{u}(0) = \mathbf{u}(1) = 0.$$



- Both components clearly features layers of width $\mathcal{O}(\varepsilon_2)$.
- u_1 also features a layer of width $\mathcal{O}(\varepsilon_1)$.

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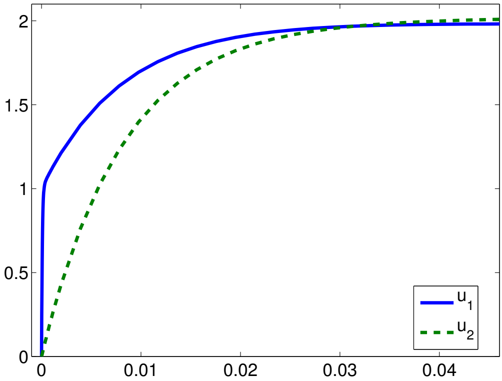

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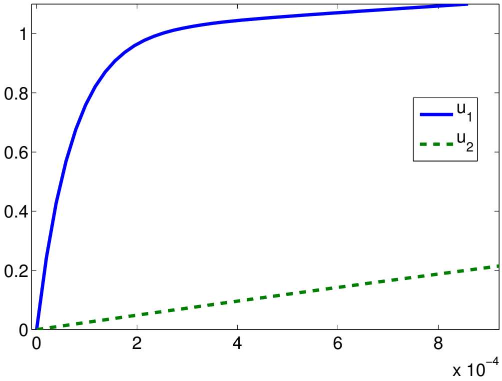

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