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A positive definite system of real-valued singularly perturbed problems

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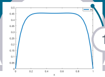
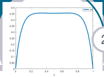
Content



Introduction



We are interested in the numerical solution of a singularly perturbed, fourth-order ordinary differential equations.

Our model differential equation is

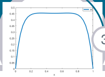
$$-\varepsilon u^{(4)}(x) + au''(x) - bu(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \qquad (1)$$

subject to the boundary conditions

$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0.$$

Here ε is a positive, real-valued parameter: $0 < \varepsilon \leq 1$, but typically $\varepsilon \ll 1$. And so the problem is **singularly perturbed**.
The coefficient functions a, b and right-hand side function f are real or complex-valued functions on the interval Ω.

Real-valued singularly perturbed problems



We consider the numerical solution of real-valued singularly perturbed, fourth-order ordinary differential equations, and in particular, problems of the following form.

[Shanthi and Ramanujam, 2002]:

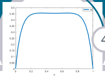
$$-\varepsilon u^{(4)}(x) + au''(x) - bu(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \quad (2)$$

subject to the boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

The coefficient functions a, b and right-hand side function f are real-valued functions on the interval Ω.

A positive definite system



We now propose an simple approach that allows one to reformulate (2) as a system with a (non-symmetric) positive definite coefficient matrix. We set

$$u'' := \alpha w + \beta u,$$

and, consequently,

$$u^{(4)} = \alpha w'' + \beta\alpha w + \beta^2 u.$$

With this, (2) can be transformed as a system of two equations of the form

$$-\varepsilon w'' + (a\alpha - \varepsilon\alpha\beta)w + (a\beta - \varepsilon\beta^2 - b)u = f, \tag{3a}$$

$$-u'' + \alpha w + \beta u = 0, \tag{3b}$$

subject to the boundary conditions

$$u(0) = w(0) = u(1) = w(1) = 0.$$

Positive definite matrix

Written in matrix form, this is

$$-\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w'' \\ u'' \end{pmatrix} + B \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

where

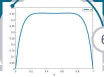
$$B = \begin{pmatrix} a\alpha - \varepsilon\alpha\beta & a\beta - \varepsilon\beta^2 - b \\ \alpha & \beta \end{pmatrix}.$$

This B satisfies $\mathbf{v}^T B\mathbf{v} \geq \gamma \mathbf{v}^T\mathbf{v}$, for all v, if and only if, $M = (B^T + B)/2$ is symmetric positive definite. Here

$$M = \begin{pmatrix} a\alpha - \varepsilon\alpha\beta & \frac{1}{2}(a\beta - \varepsilon\beta^2 - b + \alpha) \\ \frac{1}{2}(a\beta - \varepsilon\beta^2 - b + \alpha) & \beta \end{pmatrix}.$$

Clearly, M is symmetric.

Tests for Positive definite matrix M



What conditions on α and β ensure that M is a positive definite?

Each of the following tests is a necessary and sufficient condition for a symmetric matrix M to be Positive definite:

(i) $\mathbf{v}^T M \mathbf{v} \geq \gamma \mathbf{v}^T \mathbf{v}$, for all v,

(ii) All eigenvalues of M are positive,

(iii) Determinant test,

(iv) Pivot test.

There is the case if M is *strictly diagonally dominant*, with positive diagonal entries, i.e., $M_{ii} > \sum_{j \neq i} |M_{ij}|$ for $i = 1, 2$ [Beezer, 2008]. So, thus, we require that

(i) $|a\alpha - \varepsilon\alpha\beta| > 0$,

(ii) $|\beta| > 0$,

(iii) $|a\alpha - \varepsilon\alpha\beta| > \frac{1}{2}|a\beta - \varepsilon\beta^2 - b + \alpha|$,

(iv) $|\beta| > \frac{1}{2}|a\beta - \varepsilon\beta^2 - b + \alpha|$.

Eigenvalue test

By applying the eigenvalue test on the matrix M. We can see that

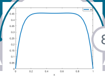
$$-\varepsilon\beta^2 + a\beta + b - 2\sqrt{a\beta b - \varepsilon\beta^2 b} \leq \alpha \leq -\varepsilon\beta^2 + a\beta + b + 2\sqrt{a\beta b - \varepsilon\beta^2 b}$$

Suppose $\beta = 1$, then $\alpha = a + b - \varepsilon$. These are two conditions on α and β to be ensure that M is a positive definite. We can rewrite (2) as a system

$$-\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w'' \\ u'' \end{pmatrix} + B \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

where

$$B = \begin{pmatrix} (a + b - \varepsilon)(a - \varepsilon) & a - \varepsilon - b \\ a + b - \varepsilon & 1 \end{pmatrix},$$



Coercivity property of the matrix

Lemma

When $\beta = 1$ *and* $\alpha - a + b - \varepsilon$*, the matrix M is coercive (for sufficiently small* ε*). Further more there is a positive* γ *such that* $\gamma \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$*, where* λ_{min} *and* λ_{max} *are eigenvalues of matrix M, and*

$$\frac{\mathbf{v}^T M \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \geq \gamma \quad \text{for all } \mathbf{v} \in \mathbb{R}^2. \tag{4}$$

Example

Now consider the real-valued problem:

$$-\varepsilon u^{(4)}(x) + 2u''(x) - 4u(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \quad (5)$$

subject to the boundary conditions

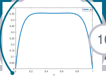
$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0.$$

If we take $\alpha = 6 - \varepsilon$ and $\beta = 1$, we have

$$B = \begin{pmatrix} 12 - 8\varepsilon + \varepsilon^2 & -(\varepsilon + 2) \\ 6 - \varepsilon & 1 \end{pmatrix}, \; \textit{and} \;\; M = \begin{pmatrix} 12 - 8\varepsilon + \varepsilon^2 & -\varepsilon + 2 \\ -\varepsilon + 2 & 1 \end{pmatrix}.$$

where M is a symmetric positive definite for $\varepsilon < 1$.

Conclusions and future work



- We also aim to extend the work to complex-valued case where the associated 4×4 coefficient matrix in the second order system is a positive definite.
- We are now working on the analysis of methods for fourth-order complex-valued problems in the case where the problem can be re-cast as a coupled system of second-order problems (see, e.g., [Xenophontos et al., 2013]).
- We also aim to extend the work to complex-valued fourth-order ones which cannot be written as a coupled system of second-order ones (e.g., [Constantinou et al., 2016]).

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Thank you