Internal-variable modelling of solids with slow dynamics

Harold Berjamin[1], Guillaume Chiavassa[2],
Nicolas Favrie[3], Bruno Lombard[1], Emmanuelle Sarrouy[1]

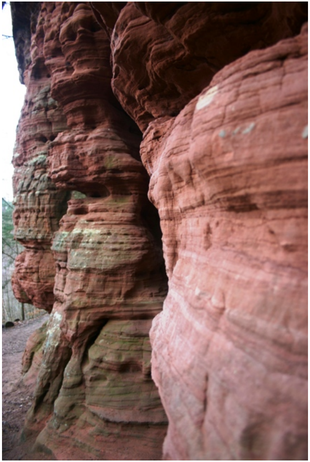
[1]Laboratoire de Mécanique et d'Acoustique, Marseille (France)
berjamin@lma.cnrs-mrs.fr
[2]Laboratoire de Mécanique, Modélisation et Procédés Propres, Marseille (France)
[3]Institut Universitaire des Systèmes Thermiques Industriels, Marseille (France)

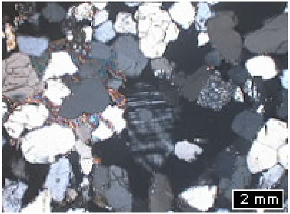
Modelling group seminar, NUI Galway, September 28, 2018

Who I am

26, French and German
École Centrale de Marseille: *general engineering*, acoustics
PhD viva: 29th November 2018

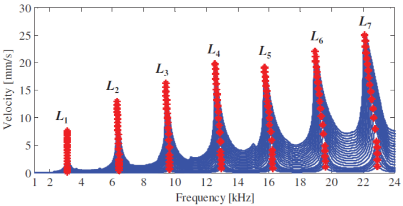
Sandstone across the scales

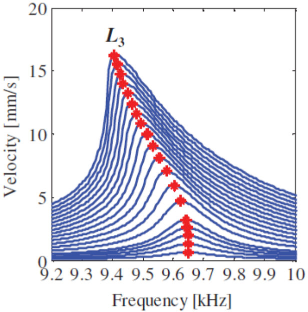




Forced longitudinal vibrations (1)

Resonance (NRUS) *Johnson 96, Remillieux 16*



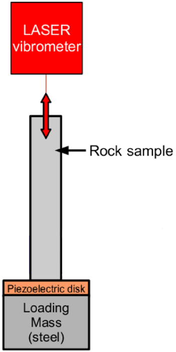


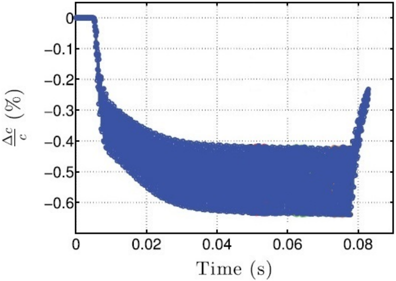
Measurement of *steady-state frequency* response

- low strain $\varepsilon \sim 10^{-8} \rightarrow 10^{-6}$
- softening ($\simeq$ Duffing *TenCate et al. 04*)
- harmonic generation

Forced longitudinal vibrations (2)

Dynamic Acoustoelasticity (DAE) *Renaud et al. 12, Rivière et al. 13*



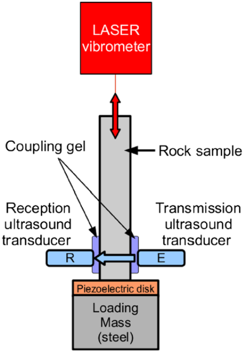


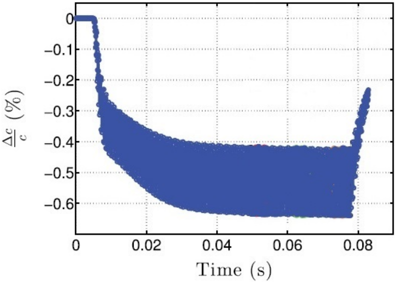
Local measurement over *time*

- softening/recovery *transients* ("slow dynamics")
- no permanent damage

Forced longitudinal vibrations (2)

Dynamic Acoustoelasticity (DAE) *Renaud et al. 12, Rivière et al. 13*



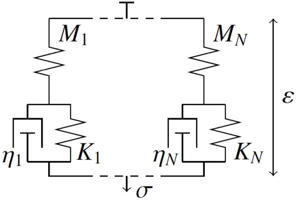


Local measurement over *time*

- softening/recovery *transients* ("slow dynamics")
- no permanent damage

Contribution

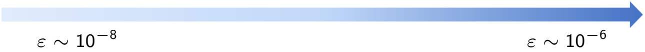
Generalized Zener rheology



Softening (internal variable)

$$\sigma = (1 - g)E\varepsilon, \qquad \varepsilon = \partial_x u$$

$$\tau \dot{g} = \tfrac{1}{2}E\varepsilon^2 - \gamma g$$



Outline

1. Modelling
2. Time-domain numerical method
3. Frequency-domain numerical method

Outline

Thermodynamics with internal variable

Adiabatic transformation
Variables of state $\{s, \varepsilon, g\}$ (elasticity + internal variable)

Clausius–Duhem inequality

$$\left(\sigma - \frac{\partial U}{\partial \varepsilon}\right)\dot{\varepsilon} - \frac{\partial U}{\partial g}\dot{g} \geq 0 \qquad \text{for all} \qquad \{\varepsilon, g, \dot{\varepsilon}\}$$

$$\Longrightarrow \qquad \sigma = \frac{\partial U}{\partial \varepsilon} \qquad \text{and} \qquad -\frac{\partial U}{\partial g}\dot{g} \geq 0$$

Thermodynamics with internal variable

Adiabatic transformation
Variables of state $\{s, \varepsilon, g\}$ (elasticity + internal variable)

Clausius–Duhem inequality

$$\left(\sigma - \frac{\partial U}{\partial \varepsilon}\right)\dot{\varepsilon} - \frac{\partial U}{\partial g}\dot{g} \geq 0 \qquad \text{for all} \qquad \{\varepsilon, g, \dot{\varepsilon}\}$$

$$\Longrightarrow \quad \boxed{\sigma = \frac{\partial U}{\partial \varepsilon}} \qquad \text{and} \qquad -\frac{\partial U}{\partial g}\dot{g} \geq 0$$

If $\boxed{\sigma = \phi_1(g) W'(\varepsilon)}$, then

$$\boxed{U = \phi_1(g) W(\varepsilon) + \phi_2(g)} \qquad \text{and} \qquad -\left(\phi_1'(g) W(\varepsilon) + \phi_2'(g)\right)\dot{g} \geq 0$$

Choice: $\tau \dot{g} = -\left(\phi_1'(g) W(\varepsilon) + \phi_2'(g)\right)$ with $\tau > 0$

Thermodynamics with internal variable

Adiabatic transformation
Variables of state $\{s, \varepsilon, g\}$ (elasticity + internal variable)

Clausius–Duhem inequality

$$\left(\sigma - \frac{\partial U}{\partial \varepsilon}\right)\dot{\varepsilon} - \frac{\partial U}{\partial g}\dot{g} \geq 0 \qquad \text{for all} \qquad \{\varepsilon, g, \dot{\varepsilon}\}$$

$$\Longrightarrow \qquad \sigma = \frac{\partial U}{\partial \varepsilon} \qquad \text{and} \qquad -\frac{\partial U}{\partial g}\dot{g} \geq 0$$

If $\sigma = \phi_1(g) W'(\varepsilon)$, then

$$U = \phi_1(g) W(\varepsilon) + \phi_2(g) \qquad \text{and} \qquad -(\phi_1'(g) W(\varepsilon) + \phi_2'(g))\dot{g} \geq 0$$

Choice: $\tau \dot{g} = -(\phi_1'(g) W(\varepsilon) + \phi_2'(g))$ with $\tau > 0$

Thermodynamics with internal variable

Adiabatic transformation

Variables of state $\{s, \varepsilon, g\}$ (elasticity + internal variable)

Clausius–Duhem inequality

$$\left(\sigma - \frac{\partial U}{\partial \varepsilon}\right)\dot{\varepsilon} - \frac{\partial U}{\partial g}\dot{g} \geq 0 \qquad \text{for all} \qquad \{\varepsilon, g, \dot{\varepsilon}\}$$

$$\Longrightarrow \qquad \sigma = \frac{\partial U}{\partial \varepsilon} \qquad \text{and} \qquad -\frac{\partial U}{\partial g}\dot{g} \geq 0$$

If $\sigma = \phi_1(g)W'(\varepsilon)$, then

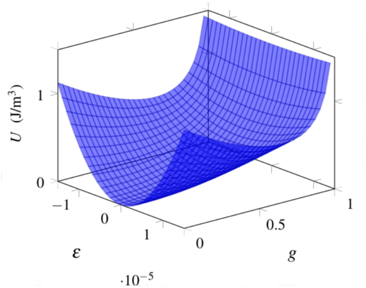
$$U = \phi_1(g)W(\varepsilon) + \phi_2(g) \qquad \text{and} \qquad -(\phi_1'(g)W(\varepsilon) + \phi_2'(g))\dot{g} \geq 0$$

Choice: $\tau\dot{g} = -(\phi_1'(g)W(\varepsilon) + \phi_2'(g))$ with $\tau > 0$

Modelling choices

From *Berjamin et al. 17*

- mathematical considerations
- qualitative behaviour



Internal energy $U = \phi_1(g)W(\varepsilon) + \phi_2(g)$
Possible choices:

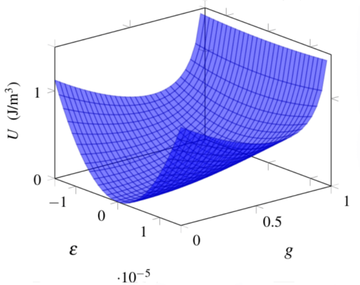
$$W(\varepsilon) \simeq \tfrac{1}{2}E\varepsilon^2$$
$$\phi_1(g) = 1 - g \quad \text{for} \quad g < 1$$
$$\phi_2(g) \simeq \tfrac{1}{2}\gamma g^2$$

$$\sigma = \phi_1(g)W'(\varepsilon)$$
$$\tau\dot{g} = -\left(\phi_1'(g)W(\varepsilon) + \phi_2'(g)\right)$$

Modelling choices

From Berjamin et al. 17

- mathematical considerations
- qualitative behaviour



Internal energy $U = \phi_1(g)W(\varepsilon) + \phi_2(g)$
Possible choices:

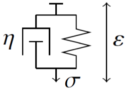
$$W(\varepsilon) \simeq \frac{1}{2}E\varepsilon^2$$
$$\phi_1(g) = 1 - g \quad \text{for } g < 1$$
$$\phi_2(g) \simeq \frac{1}{2}\gamma g^2$$

$$\sigma = (1 - g)E\varepsilon$$
$$\tau\dot{g} = \frac{1}{2}E\varepsilon^2 - \gamma g$$

Viscoelastic cases

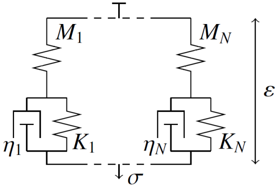
Kelvin–Voigt rheology $\{s, \varepsilon, \dot{\varepsilon}, g\}$

$$\sigma = (1-g)\,(E\varepsilon + \eta\dot{\varepsilon})$$
$$\tau\dot{g} = \tfrac{1}{2}E\varepsilon^2 - \gamma g$$



Generalized Zener rheology $\{s, \varepsilon, \varepsilon-\xi_1, \dots, \varepsilon-\xi_N, g\}$ *Berjamin et al. 18*

$$\sigma = (1-g)\sum_{\ell=1}^{N} M_\ell \xi_\ell$$
$$\tau\dot{g} = \frac{1}{2}\left(\sum_{\ell=1}^{N} M_\ell(\xi_\ell)^2 + K_\ell(\varepsilon-\xi_\ell)^2\right) - \gamma g$$
$$\eta_\ell\dot{\xi}_\ell = \eta_\ell\dot{\varepsilon} + K_\ell\varepsilon - (M_\ell+K_\ell)\xi_\ell$$



Outline

Lagrangian equations of motion

First-order strong formulation *Berjamin et al. 18*

$$\boxed{\begin{aligned}\partial_t \varepsilon &= \partial_x v\\ \rho_0 \partial_t v &= \partial_x \sigma\end{aligned}} \quad \text{and} \quad \begin{cases} \sigma = (1-g) \sum_{\ell=1}^{N} M_\ell \xi_\ell \\ \tau \partial_t g = \frac{1}{2} \left(\sum_{\ell=1}^{N} M_\ell (\xi_\ell)^2 + K_\ell (\varepsilon - \xi_\ell)^2 \right) - \gamma g \\ \eta_\ell \partial_t \xi_\ell = \eta_\ell \partial_t \varepsilon + K_\ell \varepsilon - (M_\ell + K_\ell) \xi_\ell \end{cases}$$

Balance laws $\partial_t \boldsymbol{q} + \partial_x \boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{r}(\boldsymbol{q})$, with sound speeds

$$\{-c, 0, \dots, 0, +c\} \qquad \text{where} \qquad \rho_0 c^2 = (1-g) \sum_{\ell=1}^{N} M_\ell$$

Finite-volume method *LeVeque 02*

Finite-volume method

Operator splitting *Berjamin et al. 18*

$$\partial_t \boldsymbol{q} + \partial_x \boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{r}(\boldsymbol{q}) \qquad \text{as} \qquad \begin{aligned} (\mathcal{H}_a) &: \quad \partial_t \boldsymbol{q} + \partial_x \boldsymbol{f}(\boldsymbol{q}) &= \boldsymbol{0} \\ (\mathcal{H}_b) &: \quad \partial_t \boldsymbol{q} &= \boldsymbol{r}(\boldsymbol{q}) \end{aligned}$$

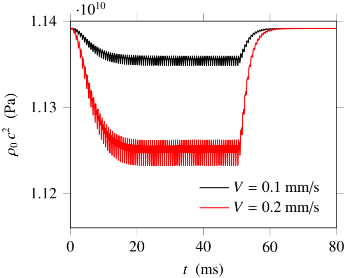
Strang splitting scheme

- $(\mathcal{H}_a)$: fourth-order ADER flux
- $(\mathcal{H}_b)$: fourth-order adaptive Rosenbrock method

Configuration

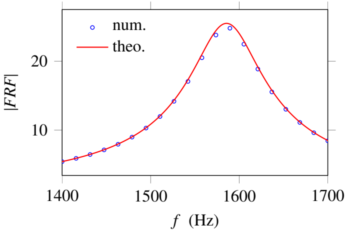
Results

Dynamic Acoustoelasticity

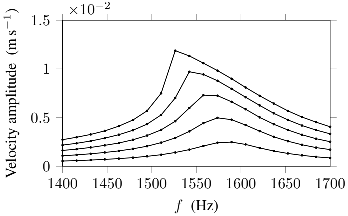


Resonance curves

Linear



Nonlinear



20 freq. $\simeq$ 40 min. $\rightarrow$

Outline

Lagrangian equations of motion

Second-order-in-time weak formulation

$$\rho_0 \langle \partial_{tt} u, \tilde{u} \rangle = (\sigma \tilde{u})|_{x=L} - (\sigma \tilde{u})|_{x=0} - \langle \sigma, \partial_x \tilde{u} \rangle$$

for all $\tilde{u} \in H^1([0, L])$, and

$$\begin{cases} \sigma = (1 - g) \sum_{\ell=1}^{N} M_\ell \xi_\ell \\ \tau \partial_t g = \frac{1}{2} \left(\sum_{\ell=1}^{N} M_\ell (\xi_\ell)^2 + K_\ell (\varepsilon - \xi_\ell)^2 \right) - \gamma g \\ \eta_\ell \partial_t \xi_\ell = \eta_\ell \partial_t \varepsilon + K_\ell \varepsilon - (M_\ell + K_\ell) \xi_\ell \end{cases}$$

Mixed finite elements (P^1-P^0) yield $\dot{\boldsymbol{X}} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{X} + \boldsymbol{Q} : (\boldsymbol{X}\boldsymbol{X}^\top)$

Harmonic balance + numerical continuation (Manlab) *Cochelin et al. 09*

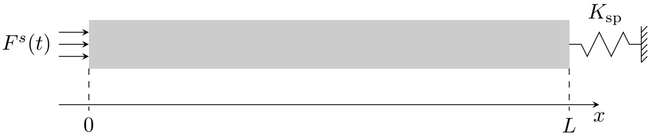
Harmonic balance + numerical continuation (Manlab)

Harmonic balance of $\dot{\boldsymbol{X}} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{X} + \boldsymbol{Q} : \left(\boldsymbol{X}\boldsymbol{X}^\top\right)$ *Cochelin et al. 09*

$$\boldsymbol{X} = \frac{\boldsymbol{a}_0}{2} + \boldsymbol{a}_1 \cos(\omega t) + \boldsymbol{b}_1 \sin(\omega t) + \cdots + \boldsymbol{b}_H \sin(H\omega t)$$

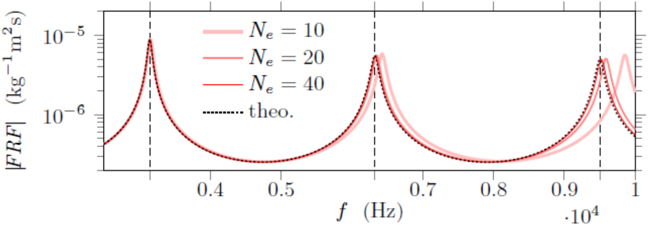
Numerical continuation: Analytic expansion of $\boldsymbol{a}_0, \ldots, \boldsymbol{b}_H$ w.r.t. parameter

Configuration

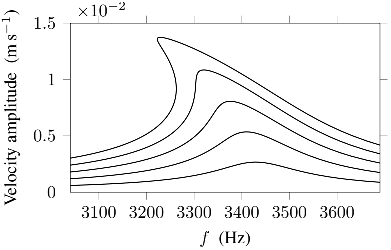


Resonance curves

Linear

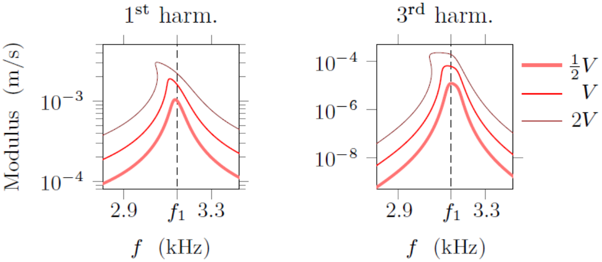


Nonlinear

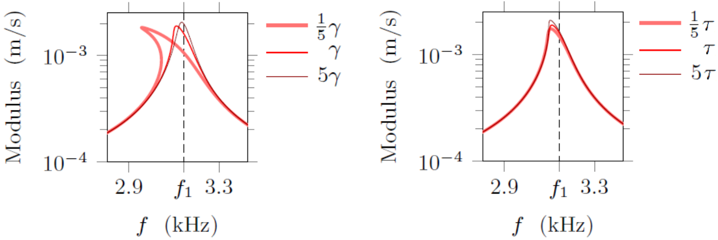


continuous freq. $\simeq$ 40 sec.$\rightarrow$

Harmonic generation



Parametric study $\tau\dot{g} = W - \gamma g$ (1st harm.)



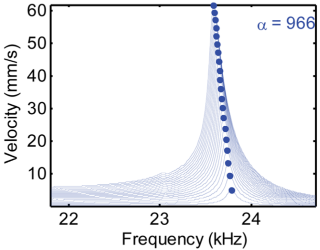
Conclusion

Rocks and concrete belong to a general class of *nonlinear viscoelastic* solids with *softening*

	response (stress BC)	response (displ. BC)	transient
time FVM	✗	(✗)	✓
freq. FEM	✓	(✓)	✗

Future works

- continuation of peaks (backbone curve)
- perturbation methods



Payan et al. 14

Multiple space dimensions (non viscous)

$$\boldsymbol{\sigma} = (1-g)\frac{1}{\det \boldsymbol{F}}\boldsymbol{F}\cdot\frac{\partial W}{\partial \boldsymbol{E}}\cdot\boldsymbol{F}^\top$$
$$\tau\dot{g} = W(\boldsymbol{E}) - \gamma g$$

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{E})^2 + \mu\operatorname{tr}\boldsymbol{E}^2 + \frac{\mathcal{C}}{3}(\operatorname{tr}\boldsymbol{E})^3 + \mathcal{B}(\operatorname{tr}\boldsymbol{E})\operatorname{tr}\boldsymbol{E}^2 + \frac{\mathcal{A}}{3}\operatorname{tr}\boldsymbol{E}^3 + \dots$$