

The Distribution of rank of Completions of Entry Pattern Matrices

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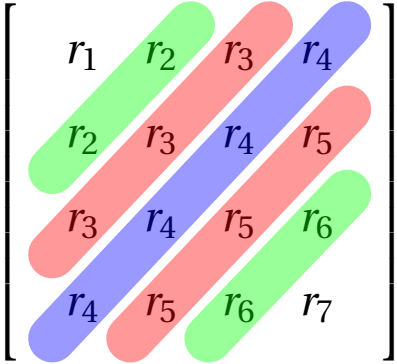
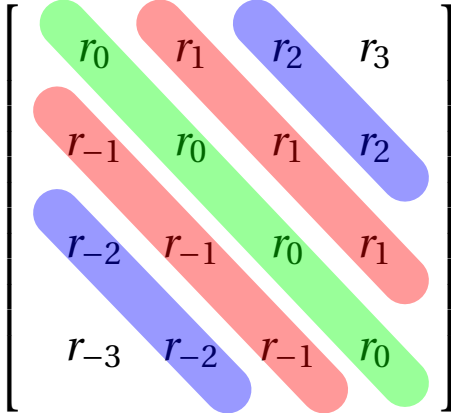
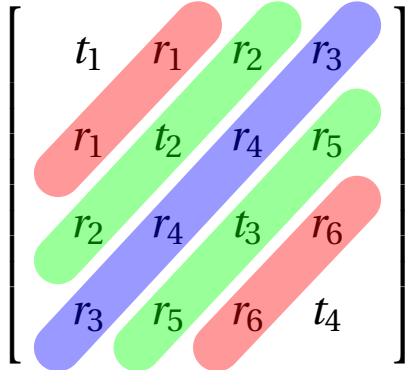
March 2, 2017



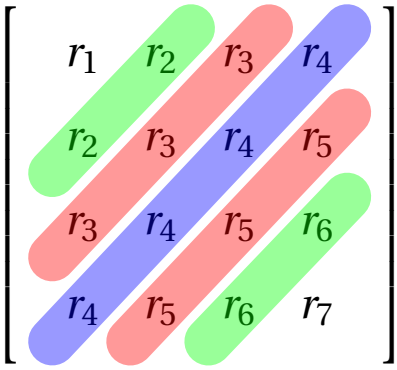
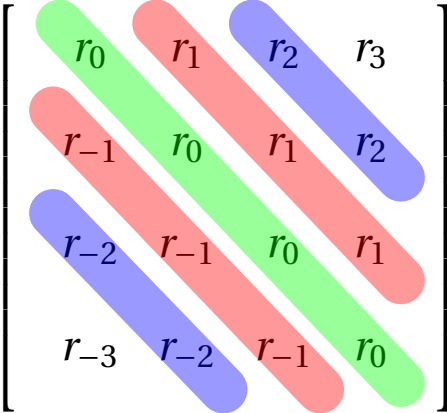
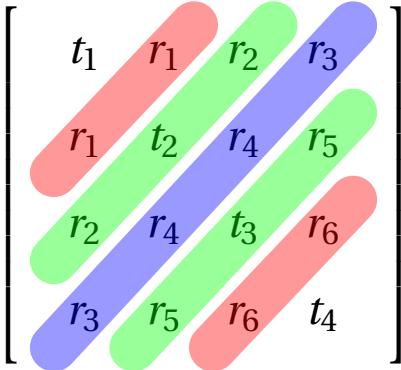
Motivation

Hankel matrix	Toeplitz matrix	symmetric matrix
$\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_3 & r_4 & r_5 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 & r_7 \end{bmatrix}$	$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_{-1} & r_0 & r_1 & r_2 \\ r_{-2} & r_{-1} & r_0 & r_1 \\ r_{-3} & r_{-2} & r_{-1} & r_0 \end{bmatrix}$	$\begin{bmatrix} t_1 & r_1 & r_2 & r_3 \\ r_1 & t_2 & r_4 & r_5 \\ r_2 & r_4 & t_3 & r_6 \\ r_3 & r_5 & r_6 & t_4 \end{bmatrix}$

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- Each entry is an element of a specified set of **independent indeterminates**.
- Entries can be **the same**, but can not be a constant, even 0.

Entry pattern matrices

In 2015, *Huang and Zhan* ([3]) noted that the above matrices have defining properties that can be expressed in term of entry pattern. → **Entry pattern matrices.**

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$\begin{bmatrix} x & y \\ x & z \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}$	$\begin{bmatrix} x & 0 \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ 2x & y \end{bmatrix}$

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*This talk is for **rank of completions** of an entry pattern matrix (EPM for short) over some fields \mathbb{F} .*

The maximum \mathbb{F} -rank

$$m_{\mathbb{F}}\text{-rank}(A) := \max_{a_1, \dots, a_k \in \mathbb{F}} \text{rank} A(a_1, \dots, a_k).$$

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Example

$$\text{Let } A(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_3 & r_4 & r_5 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 & r_7 \end{bmatrix}.$$

Then $m_{\mathbb{F}}\text{-rank}(A) = 4$ for all \mathbb{F} .

The generic \mathbb{F} -rank

$$g_{\mathbb{F}\text{-rank}}(A) := \text{rank}_{\mathbb{F}(x_1, \dots, x_k)} A$$

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We say that a square EPM $A \in M_n(S)$ is \mathbb{F} -nonsingular if $g_{\mathbb{F}}\text{-rank}(A) = n$.

The generic \mathbb{F} -rank

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$$A(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_3 & r_4 & r_5 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 & r_7 \end{bmatrix}.$$

Then $\det A$ is a non-zero polynomial in the function field $\mathbb{F}(r_1, \dots, r_7)$.

Hence, $\mathfrak{g}_{\mathbb{F}}\text{-rank}(A) = 4$ for any field \mathbb{F} .

Facts on Entry pattern matrices' ranks

- The maximum \mathbb{F} -rank can not exceed the generic \mathbb{F} -rank.

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Facts on Entry pattern matrices' ranks

- The maximum \mathbb{F} -rank can not exceed the generic \mathbb{F} -rank.
- $A(x_1, \dots, x_k) \in M_n(x_1, \dots, x_k) \Rightarrow$ its determinant is a homogeneous polynomial of degree n .
- We may restrict our attention to square EPMs. Because if

$$g_{\mathbb{F}\text{-rank}}(A) = g, m_{\mathbb{F}\text{-rank}}(A) = m$$

then there exists a $g \times g$ submatrix \mathfrak{a} of A such that

$$g_{\mathbb{F}\text{-rank}}(\mathfrak{a}) = g, m_{\mathbb{F}\text{-rank}}(\mathfrak{a}) = m.$$

$$\begin{bmatrix} x & y & z \\ y & x & z \\ y & x & z \end{bmatrix} \mapsto \begin{bmatrix} x & y & z \\ y & x & z \\ \cancel{y} & \cancel{x} & \cancel{z} \end{bmatrix}$$

Question???



- How do the maximum \mathbb{F} -rank and generic \mathbb{F} -rank depend on ground field \mathbb{F} ?
- How do the ranks depend on the number of indeterminates?

Large fields

Let $A(x_1, x_2, \dots, x_k)$ be an entry pattern matrix with generic \mathbb{F} -rank r . If the field \mathbb{F} has at least r elements then the maximum \mathbb{F} -rank is equal to the generic \mathbb{F} -rank.

$$|\mathbb{F}| \geq \text{g}_{\mathbb{F}}\text{-rank}(A) \Rightarrow \text{m}_{\mathbb{F}}\text{-rank}(A) = \text{g}_{\mathbb{F}}\text{-rank}(A).$$

Let $A(x_1, x_2, \dots, x_k)$ be an entry pattern matrix with generic \mathbb{F} -rank r . If the field \mathbb{F} has at least r elements then the maximum \mathbb{F} -rank is equal to the generic \mathbb{F} -rank.

$$|\mathbb{F}| \geq \text{g}_{\mathbb{F}}\text{-rank}(A) \Rightarrow \text{m}_{\mathbb{F}}\text{-rank}(A) = \text{g}_{\mathbb{F}}\text{-rank}(A).$$

- In particular, if the characteristic of \mathbb{F} is 0, then the maximum \mathbb{F} -rank is equal to the generic \mathbb{F} -rank of any EPM.

Small number of indeterminates

If the number of indeterminates $k < 3$, then the maximum \mathbb{F} -rank and the generic \mathbb{F} -rank of A coincide for every ground field \mathbb{F} .

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- If the number of indeterminates $k = 1$, then

$$\text{m}_{\mathbb{F}}\text{-rank}(A) = \text{g}_{\mathbb{F}}\text{-rank}(A) = 1.$$

- If A is a $r \times r$ \mathbb{F} -nonsingular EPM having 2 indeterminates x, y ; then

$$\det A = (x - y)^{r-1}(\alpha x + \beta y)$$

for some $0 \neq (\alpha, \beta)$. Thus,

$$\text{g}_{\mathbb{F}}\text{-rank}(A) = \text{m}_{\mathbb{F}}\text{-rank}(A) = r.$$

EPM-rank-tight fields

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We say that the field \mathbb{F}_q is *EPM-rank-tight* if there exists a $(q+1) \times (q+1)$ EPM whose generic \mathbb{F}_q -rank is equal to $q+1$ and exceeds its maximum \mathbb{F}_q -rank.

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Example

Let

$$B(x, y, z) = \begin{bmatrix} x & y & y & y \\ y & x & z & z \\ z & z & x & x \\ y & y & z & y \end{bmatrix}.$$

Then

$$\det B = (x-y)(x-z)(y-z)(x+y+z) \Rightarrow \mathfrak{g}_{\mathbb{F}_3}\text{-rank}(B) = 4.$$

But $\det B(x, y, z) = 0$ for all $x, y, z \in \mathbb{F}_3$ and $\text{rank } B(1, 0, 0) = 3$, hence $\mathfrak{m}_{\mathbb{F}_3}\text{-rank}(B) = 3$.

Our goal

- Any finite extensions of EPM-rank-tight field is also EPM-rank-tight **(extension theorem)**.
- $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}$ are EPM-rank-tight field.

Our goal

- Any finite extensions of EPM-rank-tight field is also EPM-rank-tight **(extension theorem)**.
- $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}$ are EPM-rank-tight field.
- Any extension of \mathbb{F}_2 (except \mathbb{F}_2) is EPM-rank-tight.
- \mathbb{F}_2 is not an EPM-rank-tight field.

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- The extension theorem still holds if we replace \mathbb{F}_p by any EPM-rank-tight field \mathbb{F}_q .

Proof of Extension theorem

\mathbb{F}_q is EPM-rank-tight $\iff \exists$ EPM $A(x, y, z) : \det A$ is a scalar product of

$$c_{q+1}(x, y, z) = xy(x^{q-1} - y^{q-1}) + yz(y^{q-1} - z^{q-1}) + zx(z^{q-1} - x^{q-1})$$

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① **Step 1.** \mathbb{F}_p is EPM-rank-tight, so $\exists A(x, y, z) \in M_{p+1}(x, y, z) : \det A$ is a scalar multiple of

$$c_{p+1}(x, y, z) = xy(x^{p-1} - y^{p-1}) + yz(y^{p-1} - z^{p-1}) + zx(z^{p-1} - x^{p-1}).$$

② **Step 2.** We may construct an EPM $A'(x, y, z)$ and a matrix B such that

$$M := \begin{bmatrix} A & A' \\ 0 & B \end{bmatrix} \sim \text{an EPM}$$

where $\det B = \frac{c_{q+1}}{c_{p+1}}$.

The field \mathbb{F}_2 is **NOT** EPM-rank-tight.

Proof.

- Suppose, contrary to our claim, that \mathbb{F}_2 is EPM-rank-tight. This means $\exists A \in M_3(x_1, \dots, x_k) : \mathfrak{g}_{\mathbb{F}_2}\text{-rank}(A) = 3 > \mathfrak{m}_{\mathbb{F}_2}\text{-rank}(A)$.
- The order of matrix A is small, so just check all cases.

Theorem

If $q = 2^k$, then the field \mathbb{F}_q is EPM-rank-tight if and only if $k \geq 2$.

Proof for \mathbb{F}_8 .

$$c_9(x, y, z) = x^8 y - xy^8 + y^8 z - yz^8 + z^8 x - zx^8 = XY(X - Y)H_6(X, Y),$$

where $X = x - z$, $Y = y - z$.

$$H_6(X, Y) = \sum_{i=0}^6 X^i Y^{6-i} = \det \begin{bmatrix} X+Y & X & 0 & \cdots & 0 \\ Y & X+Y & X & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X+Y \end{bmatrix}_{6 \times 6}$$

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$$H_6(X, Y) = \det \begin{bmatrix} X & Y & X & 0 & \cdots & 0 & 0 \\ Y & X & Y & X & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Y & X + Y \end{bmatrix}_{6 \times 6}$$

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where $X = x - z$, $Y = y - z$.

$$(X - Y)H_6(X, Y) = (X - Y) \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & X & Y & X & 0 & 0 & 0 & 0 \\ 0 & Y & X & Y & X & 0 & 0 & 0 \\ 0 & 0 & Y & X & Y & X & 0 & 0 \\ 0 & 0 & 0 & Y & X & Y & X & 0 \\ 0 & 0 & 0 & 0 & 0 & Y & X & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & Y & X + Y \end{bmatrix}_{7 \times 7}$$

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$$(X - Y)H_6(X, Y) = \det \begin{bmatrix} x-y & x-y & x-y & x-y & x-y & x-y & x-y & x-y \\ z & x & y & x & z & z & z & z \\ z & y & x & y & x & z & z & z \\ z & z & y & x & y & x & z & z \\ z & z & z & y & x & y & x & z \\ z & z & z & z & z & y & x & y \\ y & y & y & y & y & y & z & x \end{bmatrix}_{7 \times 7}$$

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$$c_9(x, y, z) = x^8 y - xy^8 + y^8 z - yz^8 + z^8 x - zx^8 = XY(X - Y)H_6(X, Y),$$

where $X = x - z$, $Y = y - z$.

$$Y(X - Y)H_6(X, Y) = \det \begin{bmatrix} y-z & y & y & y & y & y & y & y \\ 0 & x-y & x-y & x-y & x-y & x-y & x-y & x-y \\ 0 & z & y & x & y & z & z & z \\ 0 & z & x & y & x & y & z & z \\ 0 & z & z & x & y & x & y & z \\ 0 & z & z & z & x & y & x & y \\ 0 & z & z & z & z & x & y & x \\ 0 & x & x & x & x & x & z & y \end{bmatrix}_{8 \times 8}$$

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If $q = 2^k$, then the field \mathbb{F}_q is EPM-rank-tight if and only if $k \geq 2$.

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$$c_9(x, y, z) = XY(X - Y)H_6(X, Y) = \det \begin{bmatrix} x-z & 0 & 0 & 0 & \cdots & 0 \\ z & y-z & y & y & \cdots & y \\ 0 & 0 & x-y & x-y & \cdots & x-y \\ z & 0 & z & y & \cdots & z \\ z & 0 & z & x & \cdots & z \\ z & 0 & z & z & \cdots & z \\ z & 0 & z & z & \cdots & y \\ z & 0 & z & z & \cdots & x \\ z & 0 & z & z & \cdots & x \\ z & 0 & x & x & \cdots & y \end{bmatrix}_{9 \times 9}$$

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If $q = 2^k$, then the field \mathbb{F}_q is EPM-rank-tight if and only if $k \geq 2$.

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$$c_9(x, y, z) = XY(X - Y)H_6(X, Y) = \det \begin{bmatrix} x & x & z & y & x & y & z & z & z \\ z & y & y & y & y & y & y & y & y \\ z & y & x & x & x & x & x & x & x \\ z & z & z & y & x & y & z & z & z \\ z & z & z & x & y & x & y & z & z \\ z & z & z & z & x & y & x & y & z \\ z & z & z & z & z & x & y & x & y \\ z & z & z & z & z & z & x & y & x \\ z & z & x & x & x & x & x & z & y \end{bmatrix}_{9 \times 9}$$

$$\begin{aligned}
c_6(x, y, z) &= xy(x^5 - y^5) + yz(y^5 - z^5) + zx(z^5 - x^5) \\
&= C_6(X, Y) = XY(X + Y)(X + 2Y)(X + 3Y)(X + 4Y) \\
&= 0 \quad \forall x, y, z \in \mathbb{F}_5,
\end{aligned}$$

where $X = x - z$, $Y = y - z$.

Note that in $\mathbb{F}_5(x, y, z)$, we have

- $X(X + 3Y) = (X - Y)^2 - Y^2 = \begin{vmatrix} X - Y & Y \\ Y & X - Y \end{vmatrix} = \begin{vmatrix} x - y & y - z \\ y - z & x - y \end{vmatrix}$.
- $(X + Y)(X + 4Y) = X^2 - Y^2 = \begin{vmatrix} X & -Y \\ -Y & X \end{vmatrix} = \begin{vmatrix} x - z & z - y \\ z - y & x - z \end{vmatrix}$.
- $p(X, Y) = X + 2Y = x + 2y + 2z$ and $Y = y - z$ as factors omitted.

Minimal constructions for $p = 5$

$$X(X + Y)(X + 3Y)(X + 4Y) = \det \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & x-y & y-z & * & * \\ 0 & 0 & y-z & x-y & * & * \\ 0 & 0 & 0 & 0 & x-z & z-y \\ 0 & 0 & 0 & 0 & z-y & x-z \end{bmatrix}$$

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$$X(X + Y)(X + 3Y)(X + 4Y) = \det \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ y & z & x & y & * & * \\ z & y & y & x & * & * \\ z & y & z & y & x & z \\ y & z & y & z & z & x \end{bmatrix}$$

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Minimal constructions for $p = 5$

$$XY(X + Y)(X + 2Y)(X + 3Y)(X + 4Y) = \det \begin{bmatrix} y & z & y & z & y & z \\ x & x & z & z & y & y \\ y & z & x & y & y & z \\ z & y & y & x & z & y \\ z & y & z & y & x & z \\ y & z & y & z & z & x \end{bmatrix}$$

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Why the method can not solve the question for $p \geq 17$.

The method relies on the splitting of the polynomial $C_{p+1}(X, Y)$ into a pair of linear factors (at least one of which is either X , Y or $X - Y$) and a product of $\frac{p-1}{2}$ quadratic factors each of which arises as the determinant of a 2×2 matrix whose entries are drawn from $0, \pm X, \pm Y, \pm(X - Y)$. Since $C_{p+1}(X, Y)$ is the product of $p + 1$ distinct linear factors in $\mathbb{F}_p[X, Y]$, these $\frac{p-1}{2}$ quadratic determinants must be reducible as polynomials in \mathbb{F}_p , and pairwise relatively prime over \mathbb{F}_p . Moreover they can collectively include at most two of X , Y and $X - Y$ as factors.

A straightforward count reveals 19 possibilities of quadratic polynomials which arises as a multiple of determinant of 2×2 such above matrices. 12 of which are generically reducible with either X , Y or $X - Y$ as a repeated factor. At most two of these 12 can occur amongst our choice of $\frac{p-1}{2}$ pairwise relatively prime quadratic factors, since one of X , Y or $X - Y$ must be reserved for the first row. Thus the number of 2×2 determinants that can appear in our construction is bounded above by $2 + 7 = 9$, and $p - 1$ cannot exceed 18, which gives 19 as the maximum value of p to which our method could possibly apply. We now consider whether it can apply to $p = 17$ or $p = 19$.

The seven quadratic 2×2 determinants that do not (necessarily) have X , Y or $X - Y$ as factors are enumerated below; these are reducible in some characteristics and not in others.

- ① $X^2 + Y^2$
- ② $X^2 + (X - Y)^2$
- ③ $X^2 - 2Y^2 - 2XY$
- ④ $X^2 + Y^2 - XY$
- ⑤ $X^2 - Y^2 + XY$
- ⑥ $X^2 - Y^2 - XY$
- ⑦ $X^2 + Y^2 - 3XY$

A successful construction for $p = 17$ would need to involve five of the above factors, and would thus require at least five of them to be reducible over \mathbb{F}_{17} . However, only the first two are, since none of 3, 5, 12 or 14 is a square in \mathbb{F}_{17} . A solution using this method for $p = 19$ would need to involve all seven of the above factors. This is not feasible since for example the first two are irreducible in \mathbb{F}_{19} , since -1 is not a square modulo 19.