The Distribution of rank of Completions of Entry Pattern Matrices

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Motivation

Hankel matrix	Toeplitz matrix	symmetric matrix
$\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_3 & r_4 & r_5 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 & r_7 \end{bmatrix}$	$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_{-1} & r_0 & r_1 & r_2 \\ r_{-2} & r_{-1} & r_0 & r_1 \\ r_{-3} & r_{-2} & r_{-1} & r_0 \end{bmatrix}$	$\begin{bmatrix} t_1 & r_1 & r_2 & r_3 \\ r_1 & t_2 & r_4 & r_5 \\ r_2 & r_4 & t_3 & r_6 \\ r_3 & r_5 & r_6 & t_4 \end{bmatrix}$

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- Each entry is an element of a specified set of **independent indeterminates**.
- Entries can be **the same**, but can not be a constant, even 0.

Entry pattern matrices

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This talk is for ***rank of completions*** *of an entry pattern matrix (EPM for short) over some fields* $\mathbb{F}$.

The maximum $\mathbb{F}$-rank

$$\mathrm{m}_{\mathbb{F}}\text{-rank}(A) := \max_{a_1,\cdots,a_k\in\mathbb{F}} \operatorname{rank} A(a_1,\cdots,a_k).$$

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Example

Let $A(r_1,r_2,r_3,r_4,r_5,r_6,r_7) = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_3 & r_4 & r_5 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 & r_7 \end{bmatrix}$.

Then $\mathrm{m}_{\mathbb{F}}\text{-rank}(A) = 4$ for all $\mathbb{F}$.

The generic $\mathbb{F}$-rank

$$\mathrm{g}_{\mathbb{F}}\text{-rank}(A) := \operatorname*{rank}_{\mathbb{F}(x_1,\cdots,x_k)} A$$

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Then $\det A$ is a non-zero polynomial in the function field $\mathbb{F}(r_1, \cdots, r_7)$. Hence, $g_{\mathbb{F}}$-rank$(A) = 4$ for any field $\mathbb{F}$.

Facts on Entry pattern matrices' ranks

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Facts on Entry pattern matrices' ranks

- The maximum $\mathbb{F}$-rank can not exceed the generic $\mathbb{F}$-rank.
- $A(x_1, \cdots, x_k) \in M_n(x_1, \cdots, x_k) \Rightarrow$ its determinant is a homogeneous polynomial of degree n.
- We may restrict our attention to square EPMs. Because if

$$\mathrm{g}_{\mathbb{F}}\text{-rank}(A) = g, \mathrm{m}_{\mathbb{F}}\text{-rank}(A) = m$$

then there exists a $g \times g$ submatrix $\mathfrak{a}$ of A such that

$$\mathrm{g}_{\mathbb{F}}\text{-rank}(\mathfrak{a}) = g, \mathrm{m}_{\mathbb{F}}\text{-rank}(\mathfrak{a}) = m.$$

$$\begin{bmatrix} x & y & z \\ y & x & z \\ y & x & z \end{bmatrix} \mapsto \begin{bmatrix} x & y & \cancel{z} \\ y & x & \cancel{z} \\ \cancel{y} & \cancel{x} & \cancel{z} \end{bmatrix}$$

Question???

- How do the maximum $\mathbb{F}$-rank and generic $\mathbb{F}$-rank depend on ground field $\mathbb{F}$?
- How do the ranks depend on the number of indeterminates?

Large fields

Let $A(x_1, x_2, \cdots, x_k)$ be an entry pattern matrix with generic $\mathbb{F}$-rank r. If the field $\mathbb{F}$ has at least r elements then the maximum $\mathbb{F}$-rank is equal to the generic $\mathbb{F}$-rank.

$$|\mathbb{F}| \geq \mathrm{g}_{\mathbb{F}}\text{-rank}(A) \Rightarrow \mathrm{m}_{\mathbb{F}}\text{-rank}(A) = \mathrm{g}_{\mathbb{F}}\text{-rank}(A).$$

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$$|\mathbb{F}| \geq \mathrm{g}_{\mathbb{F}}\text{-rank}(A) \Rightarrow \mathrm{m}_{\mathbb{F}}\text{-rank}(A) = \mathrm{g}_{\mathbb{F}}\text{-rank}(A).$$

- In particular, if the characteristic of $\mathbb{F}$ is 0, then the maximum $\mathbb{F}$-rank is equal to the generic $\mathbb{F}$-rank of any EPM.

Small number of indeterminates

If the number of indeterminates $k < 3$, then the maximum $\mathbb{F}$-rank and the generic $\mathbb{F}$-rank of A coincide for every ground field $\mathbb{F}$.

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- If the number of indeterminates $k = 1$, then

$$\mathrm{m}_{\mathbb{F}}\text{-rank}(A) = \mathrm{g}_{\mathbb{F}}\text{-rank}(A) = 1.$$

- If A is a $r \times r$ $\mathbb{F}$-nonsingular EPM having 2 indeterminates x, y; then

$$\det A = (x - y)^{r-1}(\alpha x + \beta y)$$

for some $0 \neq (\alpha, \beta)$. Thus,

$$\mathrm{g}_{\mathbb{F}}\text{-rank}(A) = \mathrm{m}_{\mathbb{F}}\text{-rank}(A) = r.$$

EPM-rank-tight fields

We say that the field $\mathbb{F}_q$ is *EPM- rank-tight* if there exists a $(q+1)\times(q+1)$ EPM whose generic $\mathbb{F}_q$-rank is equal to $q+1$ and exceeds its maximum $\mathbb{F}_q$-rank.

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Example

Let

$$B(x,y,z) = \begin{bmatrix} x & y & y & y \\ y & x & z & z \\ z & z & x & x \\ y & y & z & y \end{bmatrix}.$$

Then

$$\det B = (x-y)(x-z)(y-z)(x+y+z) \Rightarrow \mathrm{g}_{\mathbb{F}_3}\text{-rank}(B) = 4.$$

But $\det B(x,y,z) = 0$ for all $x,y,z \in \mathbb{F}_3$ and $\operatorname{rank} B(1,0,0) = 3$, hence

$$\mathrm{m}_{\mathbb{F}_3}\text{-rank}(B) = 3.$$

Our goal

- Any finite extensions of EPM-rank-tight field is also EPM-rank-tight **(extension theorem)**.
- $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}$ are EPM-rank-tight field.

Our goal

- Any finite extensions of EPM-rank-tight field is also EPM-rank-tight **(extension theorem)**.
- $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}$ are EPM-rank-tight field.
- Any extension of $\mathbb{F}_2$ (except $\mathbb{F}_2$) is EPM-rank-tight.
- $\mathbb{F}_2$ is not an EPM-rank-tight field.

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- The extension theorem still holds if we replace $\mathbb{F}_p$ by any EPM-rank-tight field $\mathbb{F}_q$.

Proof of Extension theorem

$\mathbb{F}_q$ is EPM-rank-tight $\iff \exists$ EPM $A(x,y,z)$: $\det A$ is a scalar product of

$$c_{q+1}(x,y,z) = xy(x^{q-1} - y^{q-1}) + yz(y^{q-1} - z^{q-1}) + zx(z^{q-1} - x^{q-1})$$

Proof of Extension theorem

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1. **Step 1.** $\mathbb{F}_p$ is EPM-rank-tight, so $\exists A(x,y,z) \in M_{p+1}(x,y,z)$: $\det A$ is a scalar multiple of

$$c_{p+1}(x,y,z) = xy(x^{p-1} - y^{p-1}) + yz(y^{p-1} - z^{p-1}) + zx(z^{p-1} - x^{p-1}).$$

2. **Step 2.** We may construct an EPM $A'(x,y,z)$ and a matrix B such that

$$M := \begin{bmatrix} A & A' \\ 0 & B \end{bmatrix} \sim \text{ an EPM}$$

where $\det B = \frac{c_{q+1}}{c_{p+1}}$.

The field $\mathbb{F}_2$ is NOT EPM-rank-tight.

Proof.

- Suppose, contrary to our claim, that $\mathbb{F}_2$ is EPM-rank-tight. This means $\exists A \in M_3(x_1, \cdots, x_k) : \mathrm{g}_{\mathbb{F}_2}\text{-rank}(A) = 3 > \mathrm{m}_{\mathbb{F}_2}\text{-rank}(A)$.
- The order of matrix A is small, so just check all cases.

Theorem

If $q = 2^k$, then the field $\mathbb{F}_q$ is EPM-rank-tight if and only if $k \geq 2$.

Proof for $\mathbb{F}_8$.

$$c_9(x,y,z) = x^8y - xy^8 + y^8z - yz^8 + z^8x - zx^8 = XY(X-Y)H_6(X,Y),$$

where $X = x - z$, $Y = y - z$.

$$H_6(X,Y) = \sum_{i=0}^{6} X^iY^{6-i} = \det\begin{bmatrix} X+Y & X & 0 & \cdots & 0 \\ Y & X+Y & X & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X+Y \end{bmatrix}_{6\times 6}$$

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$$H_6(X,Y) = \det \begin{bmatrix} X & Y & X & 0 & \cdots & 0 & 0 \\ Y & X & Y & X & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Y & X+Y \end{bmatrix}_{6\times 6}$$

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where $X = x - z$, $Y = y - z$.

$$(X-Y)H_6(X,Y) = (X-Y)\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & X & Y & X & 0 & 0 & 0 & 0 \\ 0 & Y & X & Y & X & 0 & 0 & 0 \\ 0 & 0 & Y & X & Y & X & 0 & 0 \\ 0 & 0 & 0 & Y & X & Y & X & 0 \\ 0 & 0 & 0 & 0 & 0 & Y & X & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & Y & X+Y \end{bmatrix}_{7\times 7}$$

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$$(X-Y)H_6(X,Y) = \det \begin{bmatrix} x-y & x-y & x-y & x-y & x-y & x-y & x-y & x-y \\ z & x & y & x & z & z & z & z \\ z & y & x & y & x & z & z & z \\ z & z & y & x & y & x & z & z \\ z & z & z & y & x & y & x & z \\ z & z & z & z & z & y & x & y \\ y & y & y & y & y & y & z & x \end{bmatrix}_{7\times 7}$$

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where $X = x - z$, $Y = y - z$.

$$Y(X-Y)H_6(X,Y) = \det \begin{bmatrix} y-z & y & y & y & y & y & y & y \\ 0 & x-y & x-y & x-y & x-y & x-y & x-y & x-y \\ 0 & z & y & x & y & z & z & z \\ 0 & z & x & y & x & y & z & z \\ 0 & z & z & x & y & x & y & z \\ 0 & z & z & z & x & y & x & y \\ 0 & z & z & z & z & x & y & x \\ 0 & x & x & x & x & x & z & y \end{bmatrix}_{8\times 8}$$

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$$c_9(x,y,z) = XY(X-Y)H_6(X,Y) = \det \begin{bmatrix} x-z & 0 & 0 & 0 & \cdots & 0 \\ z & y-z & y & y & \cdots & y \\ 0 & 0 & x-y & x-y & \cdots & x-y \\ z & 0 & z & y & \cdots & z \\ z & 0 & z & x & \cdots & z \\ z & 0 & z & z & \cdots & z \\ z & 0 & z & z & \cdots & y \\ z & 0 & z & z & \cdots & x \\ z & 0 & x & x & \cdots & y \end{bmatrix}_{9\times 9}$$

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$\mathbb{F}_5$

$$\begin{aligned} & c_6(x,y,z) = xy(x^5-y^5)+yz(y^5-z^5)+zx(z^5-x^5) \\ = \quad & C_6(X,Y) = XY(X+Y)(X+2Y)(X+3Y)(X+4Y) \\ = \quad & 0 \quad \forall x,y,z \in \mathbb{F}_5, \end{aligned}$$

where $X = x-z$, $Y = y-z$.
Note that in $\mathbb{F}_5(x,y,z)$, we have

- $X(X+3Y) = (X-Y)^2 - Y^2 = \begin{vmatrix} X-Y & Y \\ Y & X-Y \end{vmatrix} = \begin{vmatrix} x-y & y-z \\ y-z & x-y \end{vmatrix}$.
- $(X+Y)(X+4Y) = X^2 - Y^2 = \begin{vmatrix} X & -Y \\ -Y & X \end{vmatrix} = \begin{vmatrix} x-z & z-y \\ z-y & x-z \end{vmatrix}$.
- $p(X,Y) = X+2Y = x+2y+2z$ and $Y = y-z$ as factors omitted.

Minimal constructions for $p = 5$

$$X(X+Y)(X+3Y)(X+4Y) = \det \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & x-y & y-z & * & * \\ 0 & 0 & y-z & x-y & * & * \\ 0 & 0 & 0 & 0 & x-z & z-y \\ 0 & 0 & 0 & 0 & z-y & x-z \end{bmatrix}$$

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$$X(X+Y)(X+3Y)(X+4Y) = \det \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ y & z & x & y & * & * \\ z & y & y & x & * & * \\ z & y & z & y & x & z \\ y & z & y & z & z & x \end{bmatrix}$$

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Minimal constructions for $p = 5$

$$XY(X+Y)(X+2Y)(X+3Y)(X+4Y) = \det \begin{bmatrix} y & z & y & z & y & z \\ x & x & z & z & y & y \\ y & z & x & y & y & z \\ z & y & y & x & z & y \\ z & y & z & y & x & z \\ y & z & y & z & z & x \end{bmatrix}$$

References

- Hieu Ha Van, Rachel Quinlan.
  On the maximal rank of completions of entry pattern matrices.
  Accepted for publication on Linear Algebra and its applications.
- James McTigue, Rachel Quinlan. *Partial matrices whose completions all have the same rank*. Linear Algebra and its Applications Volume 438, Issue 1, 1 January 2013, Pages 348-360.
- Zejun Huang, Xingzhi Zhan. *Nonsymmetric normal entry patterns with the maximum number of distinct indeterminates*. Linear Algebra and its Applications, Volume 485, 15 November 2015, Pages 359–371.
- Zoran Z. Petrovic. *Spaces of real matrices of fixed small rank*. Linear Algebra and its Applications, Volume 431, Issue 8, 1 September 2009, Pages 1199-1207.

Why the method can not solve the question for $p \geq 17$.

The method relies on the splitting of the polynomial $C_{p+1}(X,Y)$ into a pair of linear factors (at least one of which is either X, Y or $X-Y$) and a product of $\frac{p-1}{2}$ quadratic factors each of which arises as the determinant of a 2×2 matrix whose entries are drawn from $0, \pm X, \pm Y, \pm(X-Y)$. Since $C_{p+1}(X,Y)$ is the product of $p+1$ distinct linear factors in $\mathbb{F}_p[X,Y]$, these $\frac{p-1}{2}$ quadratic determinants must be reducible as polynomials in $\mathbb{F}_p$, and pairwise relatively prime over $\mathbb{F}_p$. Moreover they can collectively include at most two of X, Y and $X-Y$ as factors.

A straightforward count reveals 19 possibilities of quadratic polynomials which arises as a multiple of determinant of 2×2 such above matrices. 12 of which are generically reducible with either X, Y or $X-Y$ as a repeated factor. At most two of these 12 can occur amongst our choice of $\frac{p-1}{2}$ pairwise relatively prime quadratic factors, since one of X, Y or $X-Y$ must be reserved for the first row. Thus the number of 2×2 determinants that can appear in our construction is bounded above by $2+7=9$, and $p-1$ cannot exceed 18, which gives 19 as the maximum value of p to which our method could possibly apply. We now consider whether it can apply to $p=17$ or $p=19$.

The seven quadratic 2×2 determinants that do not (necessarily) have X, Y or $X - Y$ as factors are enumerated below; these are reducible in some characteristics and not in others.

1. $X^2 + Y^2$
2. $X^2 + (X - Y)^2$
3. $X^2 - 2Y^2 - 2XY$
4. $X^2 + Y^2 - XY$
5. $X^2 - Y^2 + XY$
6. $X^2 - Y^2 - XY$
7. $X^2 + Y^2 - 3XY$

A successful construction for $p = 17$ would need to involve five of the above factors, and would thus require at least five of them to be reducible over $\mathbb{F}_{17}$. However, only the first two are, since none of $3, 5, 12$ or 14 is a square in $\mathbb{F}_{17}$. A solution using this method for $p = 19$ would need to involve all seven of the above factors. This is not feasible since for example the first two are irreducible in $\mathbb{F}_{19}$, since -1 is not a square modulo 19.