Modified Partial Differential Equations

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Approximating a solution

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In essence the method involves finding a function $u(x, t)$ that satisfies our finite difference scheme and building a PDE from this which is then compared to our original PDE using dissipation and propagation analysis.[2]

Convection equation

As an illustrative example, consider the following 1d convection PDE

$$v_t + av_x = 0 \tag{1}$$

subject to some initial / boundary conditions.

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$$v_t + a v_x = 0 \tag{1}$$

subject to some initial / boundary conditions. We chose the finite difference scheme to approximate a solution at the grid point $(x = k, t = n + 1)$

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{\Delta x}(u_{k+1}^n - u_k^n) \tag{2}$$

Taylor expansion

We set $u(x,t)$ to be a solution to equation (2) at the appropriate lattice points and peform a Taylor series expansion of $u(x,t)$ about the point $(k\Delta x, n\Delta t)$ to obtain

$$\begin{aligned} 0 = (u_t)_k^n + \frac{\Delta t}{2}(u_{tt})_k^n + \frac{\Delta t^2}{6}(u_{ttt})_k^n + \dots \\ + a(u_x)_k^n + \frac{a\Delta x}{2}(u_{xx})_k^n + \frac{a\Delta x^2}{6}(u_{xxx})_k^n + \dots \end{aligned} \tag{3}$$

Tidying up

Next we eliminate the time derivatives in equation (3) by differentiating (3) with respect to t, tt, x, xx and xt and systematically substitute back in. After a bit of work we can simplify equation (3) to

$$0 = (u_t)_k^n + a(u_x)_k^n + \frac{a\Delta x}{2}(1+R)(u_{xx})_k^n + \frac{a\Delta x^2}{6}(2R+1)(R+1)(u_{xxx})_k^n + \dots \quad (4)$$

where $R = \frac{a\Delta t}{\Delta x}$

Approximating a solution

Finally using the notation

$$v_1 = -\frac{a\Delta x}{2}(1+R)$$

and

$$c = \frac{a\Delta x^2}{6}(2R+1)(R+1)$$

we can write the solution to our modified PDE as [3]

$$u = \hat{u}e^{-v_1\beta^2 t}e^{i\beta(x-at+c\beta^2 t)} \tag{5}$$

We next use the modified PDE method and the dissipation/propagation analysis to look at a more difficult advection-diffusion PDE of the form

$$u_t + \beta u_x = \alpha u_{xx}, \ 0<x<1, \ 0<t \leq T, \tag{6}$$

with initial condition

$$u(x,0) = f(x), \ 0 \leq x \leq 1, \tag{7}$$

subject to boundary conditions

$$u(0,t) = g_0(t), \ 0<t \leq T, \tag{8}$$

$$u(1,t) = g_1(t), \ 0<t \leq T. \tag{9}$$

New fourth order scheme

Mehdi Dehghan [1] has proposed a fourth order scheme to solve equations (6)-(9) of the form

$$u_t = \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{10}$$

$$u_x = \frac{12s + 2c^2 - 3c - 2}{12} \frac{u_{i+2}^n}{2\Delta x} + \ldots \tag{11}$$

$$\frac{12s + 2c^2 + 3c - 2}{12} \frac{u_i^n - u_{i-2}^n}{2\Delta x} - \ldots \tag{12}$$

$$\frac{c^2 + 6s - 4}{3} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{13}$$

References

- Dehghan, M. “Weighted finite difference techniques for the one-dimensional advectiondiffusion equation”, Applied Mathematics and Computation, 147, pp. 307319, (2004)
- Jang, C. Y. , Apello, D. , Colonius, T. , Hagstrom, T. and Inkman, M. “An Analysis of Dispersion and Dissipation Properties of Hermite Methods and its Application to Direct Numerical Simulation of Jet Noise”, AIAA/CEAS Aeroacoustics Conference (33rd AIAA Aeroacoustics Conference), (2012).
- Stoer, J. “Introduction to Numerical Analysis”, New York; London: Springer-Verlag, (1993).