Genus Two Zhu Theory for Fermionic VOSAs II

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Introduction

In this talk, we will discuss a genus two analogue of the Zhu recursion formula developed by Mason, Tuite and Zuevsky for a genus one vertex operator superalgebra (VOSA), or equivalently, a VOSA version of the VOA recursion formula found by Gilroy and Tuite.

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- A Virasoro vector $\omega \in V$

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- For all u, v in V, we have:

$$(z - w)^N [Y(u, z), Y(v, w)] = 0$$

for a sufficiently large integer N, where $[,]$ is the commutator defined by:

$$[Y(u, z), Y(v, w)] = Y(u, z)Y(v, w) - (-1)^{p(u)p(v)} Y(v, w)Y(u, z)$$

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- $Y(\mathbf{1}, z) = Id_V$
- $Y(u, z)\mathbf{1} = u + O(z)$

VOSAs continued

- $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-1}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n}c$$

where c is a constant known as the central charge.

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$$\{v \in V : L(0)v = rv, r \in \mathbb{R}\}$$

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- $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$

Modular forms and Elliptic functions

We now define modular forms. A modular form is a function $f(\tau)$ on the upper-half complex plane $\mathbb{H}$ which:

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$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(z)$$

where $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$, for some non-negative integer k (called the weight of the form)

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  where $a, b, c, d, \in \mathbb{Z}$ and $ad-bc=1$, for some non-negative integer k (called the weight of the form)
- has a Fourier expansion

$$f(\tau)=\sum_{n=0}^{\infty} a(n)q^n$$

  where $q=exp(2\pi i\tau)$. This converges for $|q|<1$ (i.e. $\Im(\tau)>0)$

Modular forms and Elliptic Functions

The examples of interest here are the Eisenstein series

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n=0}^{\infty}\sigma_{k-1}(n)q^n$$

where q is as before, B_k is a Bernoulli number and $\sigma_{k-1}(n)$ is the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.
The E_k also have an alternative series representation:

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(n-1)!}\sum_{r\geq 0}\frac{r^{k-1}q^r}{1-q^r}$$

Following on from the E_k above we define:

$$P_n(z,\tau) = \frac{1}{z^n} + \sum_{k=2}^{\infty}\binom{k-1}{n-1}E_k(\tau)z^{k-n}$$

Note that there is no contribution from the odd k cases as then the E_k are trivial forms.

Twisted Functions

We can add additional parameters to these functions, which now become twisted Eisentein series and elliptic functions:

$$P_n \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \frac{1}{z^n} + (-1)^n \sum_{k=2}^{\infty} \binom{k-1}{n-1} E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) z^{k-n}$$

where

$$E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) = -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!} \sum_{r \geq 0}{}' \frac{(r+\lambda)^{k-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}}$$

$$+ \frac{(-1)^k}{(k-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{k-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}$$

where $\phi, \theta \in U(1)$, $\phi = \exp(2\pi i \lambda)$. Note that if we set $\theta, \phi = 1$ then $E_k \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\tau)$ becomes the classical Eisenstein series.

n-point Functions for VOSAs

The n-point function for a VOSA V is defined by

$$Z_V^{(1)}(g; v_1, z_1; \ldots; v_n, z_n; \tau)$$

$$= \mathrm{STr}_V(gY(q_1^{L(0)}v_1, q_1) \cdots Y(q_n^{L(0)}v_n, q_n)q^{L(0)-c/24})$$

where $g \in Aut(V)$ and $STr_V(A) = Tr_{V_{\bar{0}}}(A) - Tr_{V_{\bar{1}}}(A)$ for an operator A. It can also be naturally defined for a VOSA module M.

Zhu Recursion for VOSAs

n-point functions undergo *Zhu recursion* and can be expressed in terms of $(n-1)$-point functions:

$$
\begin{aligned}
&Z_V^{(1)}(g; v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\
&= \delta_{\phi,1}\delta_{\theta,1} STr_V(go(v) Y(v_1, q_1) \cdots Y(v_n, q_n) q^{L(0)-c/24}) \\
&+ \sum_{k=1}^{n} \sum_{m\geq 0} p(v, \boldsymbol{v_{k-1}}) \cdot P_{m+1}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z - z_k, \tau) \\
&\quad \times Z_V^{(1)}(g; v_1, z_1; \ldots; v[m]v_k, z_k; \ldots; v_n, z_n; \tau)
\end{aligned}
$$

where $gv = \theta^{-1}v$, $\phi = \exp(2\pi i wt(v))$ and
$p(v, \boldsymbol{v_{k-1}}) = (-1)^{p(v)[p(v_1)+\cdots+p(v_{k-1})]}$ for $r > 1$.

The idea is to use a sewing scheme introduced by Yamada and expanded on by Mason and Tuite to develop a genus two version of the above.

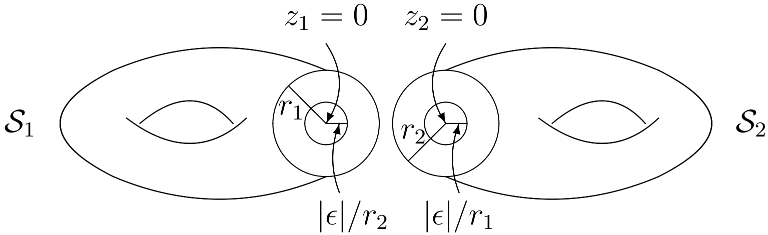

Fig. 1 Sewing Two Tori

We will refer to $\mathcal{S}_1$ and $\mathcal{S}_2$ as the left and right tori respectively.

Genus Two n-point Functions

We then build up the genus two n-point function $(n = L + R + 1)$ from genus one data:

$$Z_V^{(2)}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}, \tau_1, \tau_2, \epsilon)$$

$$= \sum_{u \in V} Z_V^{(1)}(g_1; Y[v, x]\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) Z_V^{(1)}(g_2; \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}, \tau_2)$$

where $g_1, g_2 \in \mathrm{Aut}(V)$, a_l, b_r are states and $\boldsymbol{a_l}, \boldsymbol{x_l} := a_1, x_1; \ldots; a_L, x_L$, $\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] = Y[a_1, x_1] \cdots Y[a_L, x_L]$, $\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] = Y[b_R, x_R] \cdots Y[b_1, x_1]$ and the sum is over a basis for V.

Genus Two Zhu Recursion

To develop a genus two Zhu recursion formula, we can substitute our genus one version into the left $L+1$ point function. For convenience, we will streamline notation greatly. We obtain the formula:

$$
\begin{aligned}
&Z^{(2)}_{M_1,M_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r})\\
&=\delta^{1,1}_{\theta_1,\phi_1}O_1+p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\mathbb{X}_1\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})P_{1+j}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{M_1,M_2}(\ldots;v[j]a_l,x_l;\ldots)
\end{aligned}
$$

where $p_1 = (-1)^{p(v)[p(a_1)+\ldots+p(a_L)]}$,

$$
\begin{aligned}
O_1 &= O_1(v, \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) \\
&= \sum_{u \in V} \mathrm{STr}_{M_1} \left(g_1 o(v) \boldsymbol{Y}(\boldsymbol{q_{x_l}^{L(0)}} \boldsymbol{a_l}, \boldsymbol{q_{x_l}}) Y(q_0^{L(0)} u, q_0) q_1^{L(0)-c/24} \right) \\
&\quad \times Z_{M_2}^{(1)}(g_2; \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}, \tau_2)
\end{aligned}
$$

and $\mathbb{R}(x)$, $\mathbb{X}_a$ $(a = 1, 2)$ are infinite row and column vectors (indexed from $m = 0$) given by:

$$
\mathbb{R}\begin{bmatrix}\theta \\ \phi\end{bmatrix}(x; m) = \epsilon^{\frac{m}{2}} P_{m+1}\begin{bmatrix}\theta \\ \phi\end{bmatrix}(x, \tau)
$$

$$
\mathbb{X}_1(m) = \epsilon^{-\frac{m}{2}} \sum_{u \in V} Z_{M_1}^{(1)}(g_1; \boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]v[m]u, \tau_1) Z_{M_2}^{(1)}(g_2; \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}, \tau_2)
$$

$$
\mathbb{X}_2(m) = \epsilon^{-\frac{m}{2}} \sum_{u \in V} Z_{M_1}^{(1)}(g_1; \boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) Z_{M_2}^{(1)}(g_2; \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]v[m]\overline{u}, \tau_2)
$$

The Method

The process of developing the formula is similar to that employed by Gilroy and Tuite, relating $\mathbb{X}_1$ to $\mathbb{X}_2$ (i.e., the left to the right) and using this to obtain the final Zhu reduction formula.

A Genus Two Zhu Recursion Formula

We eventually obtain the formula:

$$
\begin{aligned}
&Z^{(2)}_{M_1,M_2}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) \\
&= \delta^{1,1}_{\theta_1,\phi_1}\ {}^{N}\mathcal{F}_1 \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x) O_1 \\
&\quad + p_1 \delta^{1,1}_{\theta_2,\phi_2}\ {}^{N}\mathcal{F}_2 \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x) O_2 \\
&\quad + p_1\ {}^{N}\mathcal{F}^{\Pi} \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x) \mathbb{X}^{\delta}_1 \\
&\quad + \sum_{l=1}^{L} \sum_{j \geq 0} p(v, \boldsymbol{a_{l-1}})\ {}^{N}\mathcal{P}_{1+j} \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x, x_l) Z^{(2)}_{M_1,M_2}(\ldots; v[j]a_l, x_l; \ldots) \\
&\quad + p_1 \sum_{r=1}^{R} \sum_{j \geq 0} p(v, \boldsymbol{b_{r-1}})\ {}^{N}\mathcal{P}_{1+j} \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x, y_r) Z^{(2)}_{M_1,M_2}(\ldots; v[j]b_r, y_r; \ldots)
\end{aligned}
$$

where

$$
{}^{N}\mathcal{P}_1 \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (x, y) =
$$

$$
\begin{cases}
P_1 \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (x-y, \tau_a) + {}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\mathbb{P}_1 \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (y, \tau_a) \\
\quad - \delta^{1,1}_{\theta_a,\phi_a} \left(P_1 \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (x, \tau_a) - \pi_N \left({}^{N}\mathbb{Q}(x)\Lambda_{\bar{a}} \right) (K) \right), \quad x, y \in \widehat{\mathcal{S}}_a \\
\\
\xi^{2N} \left({}^{N}\mathbb{Q}(x)\mathbb{P}_1 \begin{bmatrix} \theta_{\bar{a}} \\ \phi_{\bar{a}} \end{bmatrix} (y, \tau_{\bar{a}}) - \delta^{1,1}_{\theta_a,\phi_a} \pi_N \left(\epsilon^{K/2} P_{K+1} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (x, \tau_a) \right.\right. \\
\quad \left.\left. - \left({}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\Lambda_a \right) (K) \right) \right), \quad x \in \widehat{\mathcal{S}}_a, y \in \widehat{\mathcal{S}}_{\bar{a}}
\end{cases}
$$

for $a = 1, 2$, $N = \mathrm{wt}(v)$, $K = 2N - 2$ and $\pi_N = 1 - \delta_{N,1} - \delta_{N,\frac{1}{2}}$.

The ξ factor is a branch cut chosen in the sewing process.

Genus Two Objects

The other objects are given by $\mathbb{X}_1^{\delta} = \Pi^{\delta}\mathbb{X}_1$, where

$$\Pi^{\delta} = \Pi - \delta^{1,1}_{\theta_1,\phi_1} E_{00} - \delta^{1,1}_{\theta_2,\phi_2} E_{KK}$$

where $\delta^{b,d}_{a,c} = \delta_{a,b}\delta_{c,d}$ is a product of Kronecker deltas, with Π an infinite projection matrix with an initial K non-trivial entries along the diagonal, and E_{ij} are infinite elementary matrices with entries given by

$$E_{ij}(m,n) = \delta^{i,j}_{m,n}$$

and

$$\begin{aligned} &O_2 = O_2(v, \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) \\ &= \sum_{u \in V} Z^{(1)}_{M_1}(g_1; \boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) \times \\ &\quad \times \mathrm{STr}_{M_2}\left(g_2 o(v) \boldsymbol{Y}(\boldsymbol{q_{y_r}^{L(0)}} \boldsymbol{a_l}, \boldsymbol{q_{y_r}}) Y(q_0^{L(0)}\overline{u}, q_0) q_2^{L(0)-c/24}\right) \end{aligned}$$

Genus Two Objects

Also

$$ {}^{N}\mathcal{P}_{1+j}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y) = \frac{1}{j!}\partial_y\left({}^{N}\mathcal{P}_{1}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)\right) $$

$$ = \begin{cases} P_{1+j}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x-y,\tau_a) + {}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\overline{a}}\mathbb{P}_{1+j}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(y,\tau_a), & x,y\in\widehat{\mathcal{S}}_a \\ \xi^{2N}\cdot {}^{N}\mathbb{Q}(x)\mathbb{P}_{1+j}\begin{bmatrix}\theta_{\overline{a}}\\ \phi_{\overline{a}}\end{bmatrix}(y,\tau_{\overline{a}}), & x\in\widehat{\mathcal{S}}_a, y\in\widehat{\mathcal{S}}_{\overline{a}} \end{cases} $$

Genus Two Objects

The boldface objects in these formulas are infinite vectors similar to those defined above (and analogous to those defined by Gilroy and Tuite), with some parameter and indexing changes:

$$\mathbb{P}_{1+j}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x;m)$$

$$= \begin{cases} (-1)^{m+1}\epsilon^{\frac{m}{2}}\binom{m+j-1}{j}\left(P_{m+j-1}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(-x,\tau)-\delta^{0,1,1}_{j,\theta,\phi}E_m(\tau)\right), \\ m\geq 1 \\ \\ 0, m=0 \end{cases}$$

for $m,j\geq 0$, $\tau\in\mathbb{H}$ and $\theta,\phi\in U(1)$. ${}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)$ is given by

$$\mathbb{R}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x)\Delta\left(\mathbb{1}-\widetilde{\Lambda}_{\bar{a}}\begin{bmatrix}\theta_{\bar{a}}\\ \phi_{\bar{a}}\end{bmatrix}\widetilde{\Lambda}_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}\right)^{-1}, \quad x\in\widehat{\mathcal{S}}_a \qquad (1)$$

Genus Two Objects

The infinite matrices $\Lambda \begin{bmatrix} \theta \\ \phi \end{bmatrix}$, $a = 1, 2$ are described by

$$\Lambda \begin{bmatrix} \theta \\ \phi \end{bmatrix} (m, n) = \begin{cases} \epsilon^{(m+n)/2} (-1)^{n+1} \binom{m+n-1}{n} E_{m+n} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) & , m \geq 1 \\ 0 & , m = 0 \end{cases}$$

Genus Two Objects

We believe that the ${}^{N}\mathcal{F}_a \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (x)$, ${}^{N}\mathcal{F}^{\Pi} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (x)$ objects are related to the space of forms living on the genus two Riemann surface.

The ${}^{N}\mathcal{F}^{\Pi}$ object is nontrivially either a $(K-1)$-, K- or $(K+1)$-dimensional vector, depending on the various values of θ_a and ϕ_a. These objects seem to conspire so that the dimension of this space is always preserved, in compliance with the Riemann-Roch theorem.

Next time

Sewing g handles to a sphere, the Schottky group etc....

References I

- Gilroy, T., Tuite, M.P.: Genus Two Zhu Theory for Vertex Operator Algebras, arXiv:1511.07664
- Mason, G. and Tuite, M.P.: On genus two Riemann surfaces formed from sewn tori. Commun. Math. Phys. **270**, 587-634 (2007).
- Mason, G., Tuite, M.P. and Zuevsky, A.: Torus npoint functions for $\mathbb{R}$-graded vertex operator superalgebras and continuous fermion orbifolds, Commun. Math. Phys. **283** (2008) 305-342.
- Serre, J-P.: A Course in Arithmetic, Springer-Verlag (Berlin 1978)

References II

- Tuite, M.P and Zuevsky, A: Genus Two Partition and Correlation Functions for Fermionic Vertex Operator Superalgebras I, Commun. Math. Phys. **306**, 419447 (2011)
- Yamada, A.: Precise variational formulas for abelian differentials. Kodai. Math. J. **3**, 114-143 (1980).
- Zhu, Y.: Modular invariance of characters of vertex operator algebras. J.Amer.Math.Soc. **9** (1996) 237-302