n-point Functions for Genus One Bosonic VOAs I

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Introduction

In this talk we will discuss work towards discussing genus one Zhu recursion on free bosonic vertex operator algebras, examining a paper of Mason and Tuite.

VOAs Revisited

Recall the (ring-theoretic) commutator $[\cdot, \cdot] : R \times R \to R$ with $[A, B] = AB - BA$ for $A, B \in R$. We will consider this in the context of the ring of linear operators $End(V)$ acting on a vector space V. $End(V)$ is also a vector space, and with $[\cdot, \cdot]$ forms a Lie algebra.

VOAs Revisited

We first recall the definition of a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$:

- A *space of states* V
- A map $Y : V \to End(V)[[z, z^{-1}]]$ which takes $v \in V$ to $\sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$
- A (unique nonzero) *vacuum vector* $\mathbf{1} \in V$ with $Y(\mathbf{1}, z) = Id_V$ and $Y(u, z)\mathbf{1} = u + \mathcal{O}(z)$ for all $u \in V$
- A Virasoro vector $\omega \in V$ with $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ where the $L(n)$ operators satisfy the Virasoro algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n}c$$

where c is a constant known as the *central charge*.

VOAs Revisited

This data satisfies the following axioms:

- Each vector has an integral eigenvalue for the operator $L(0)$ (known as the *weight*) which puts the vector into a weight space V_n. We then have

  $$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

  where the V_n all have finite dimension.
- $[L(-1), Y(v,z)] = \partial_z Y(v,z)$
- There exists an integer N such that:

  $$(w-z)^N [Y(u,w), Y(v,z)] = 0$$

  for N sufficiently large. These operators are then said to be *local of order N*.

VOAs Revisited

The bracket on the vertex operators is defined:

$$[Y(u,w), Y(v,z)] = \left[\sum_{m\in\mathbb{Z}} u(m)w^{-m-1}, \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}\right]$$

$$= \sum_{m,n\in\mathbb{Z}} [u(m), v(n)]w^{-m-1}z^{-n-1}$$

Modular Forms

Recall also the modular group

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

with the usual multiplication. This group acts on the complex upper half plane $\mathbb{H}$ as follows:

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tau \in \mathbb{H}$.

Modular Forms

Following this we define a modular form. A modular form is a function f on $\mathbb{H}$ which:

- is holomorphic on $\mathbb{H}$
- satisfies $f(\gamma \cdot \tau) = (cz + d)^k f(\tau)$ where γ is as above and k is a non-negative even integer known as the *weight* of the form
- has a Fourier expansion $f(\tau) = \sum_{n \geq 0} a_n q^n$ where $q = exp(2\pi i \tau)$.

These Fourier coefficients are of number-theoretic interest.
The modular forms we will encounter here are the Eisenstein series

$$E_k(\tau) = \frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function and B_k is the kth Bernoulli number.

The Dedekind Eta function

Another function with modular properties is the Dedekind eta function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

This function will appear across our studies of the the n-point functions of the Heisenberg VOA. It is not exactly a modular form but a modular function of weight $\frac{1}{2}$.

Elliptic Functions

We will now discuss elliptic functions. The main functions we will be concerned with here are:

$$P_1(z,\tau) = \frac{1}{z} + \sum_{k=2}^{\infty} E_k(\tau) z^{k-1}$$

and its successive derivatives:

$$P_n(z,\tau) = \frac{(-1)^n}{(n-1)!}\partial_z P_0(z,\tau) = \frac{1}{z^n} + \sum_{k=2}^{\infty} \binom{n-1}{k-1} E_k(\tau) z^{n-k}$$

These will feature in our discussion of Zhu recursion.

The Square Bracket

Zhu also defined a coordinate change to an isomorphic VOA, known as the *square bracket formalism*, defined by:

$$(V, Y(,), \mathbf{1}, \omega) \to (V, Y[,], \mathbf{1}, \widetilde{\omega})$$

where

$$Y[u, z] = \sum_{n \in \mathbb{Z}} u[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1)$$

where $q_z = \exp(z)$ and

$$\widetilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$$

This space is also integer graded (by $L[0]$) but the grading itself is different.

n-point functions

We define an n-point function on the torus as

$$Z_V^{(1)}(v_1, z_1, v_2, z_2, \ldots, v_n, z_n)$$

$$= Tr_V(Y(q_1^{L(0)}v_1, q_1)Y(q_2^{L(0)}v_2, q_2)\cdots Y(q_n^{L(0)}v_n, q_n)q^{L(0)-\frac{c}{24}})$$

following Zhu, where $q_i = e^{z_i}$, $q_i^{L(0)} = \sum_{k\geq 0}\frac{(z_i L(0))^k}{k!}$. The $q^{L(0)-\frac{c}{24}}$ factor is present to enhance the modular properties of the function. Letting $v_i = \mathbf{1}$ for all i we get

$$Z_V(\tau) = Tr_V(Id_V^n q^{L(0)-\frac{c}{24]}}) = Tr_V(q^{L(0)-\frac{c}{24}})$$

as we know $\mathbf{1} \in V_0$.

A Combinatorial Approach

Recall the rank one Heisenberg VOA M with generator a. We can then write any vector $v \in M$ as

$$v = a[-1]^{e_1} \cdots a[-p]^{e_p} \mathbf{1}$$

We then have a *labelled set*

$$\Phi_i = \{\underbrace{1, 1, \ldots, 1}_{e_1}, \underbrace{2, \ldots, 2}_{e_2}, \ldots, \underbrace{p \ldots, p}_{e_p}\}$$

corresponding to each $v_i \in M$. Then for an n-point function we consider the (disjoint) union $\Phi = \bigcup_{i=1}^{n} \Phi_i$ which is also a labelled set.

Fixed Point Free Involutions

On an arbitrary finite set S we can define the group of permutations $\Sigma(S)$. A subset of this group is the set of involutions, i.e. permutations that can be written uniquely as a product of disjoint 1- and 2-cycles. If we look at the set of permutations with no 1-cycles we get the set of *fixed-point free involutions* of $F(\Phi)$. This will appear several times in our discussion of Zhu reduction.

Next Time...

We will bring these notions together to examine some n-point functions and the Zhu reduction on these functions.

References

- Mason, G. and Tuite, M.P.: Torus chiral n-point functions for free boson and lattice vertex operator algebras, Commun.Math.Phys. 235 (2003) 47-68.
- Serre, J-P.: A Course in Arithmetic, Springer-Verlag (Berlin 1978)
- Zhu, Y.: Modular invariance of characters of vertex operator algebras. J.Amer.Math.Soc. 9 (1996) 237-302