Vertex Operator Superalgebras

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Introduction

In this talk we will discuss vertex operator superalgebras and consider some basic examples.

Superspaces

We first define the notion of a *superspace*. This is a vector space V where every vector $v \in V$ has a parity $p(v) \in \mathbb{Z}/2\mathbb{Z}$. Then we have that $V = V_{\bar{0}} \oplus V_{\bar{1}}$ where V_a is given by:

$$V_a = \{v \in V : p(v) = a\}$$

Following this we can define a vertex operator superalgebra:

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$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \in End(V)[[z, z^{-1}]]$$

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- A (unique nonzero) *vacuum vector* $\mathbf{1} \in V$ with $Y(\mathbf{1}, z) = Id_V$

VOSAs contd.

- A *Virasoro vector* $\omega \in V$ with $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ where the $L(n)$ operators satisfy the Virasoro algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m,-n}c$$

where c is a constant known as the *central charge*. where the commutator $[A, B]$ is defined as

$$[A, B] = AB - (-1)^{p(A)p(B)}BA$$

and for an operator $v(n)$ we have $p(v(n)) = p(v)$, i.e. for $p(A) = 0$ or $p(B) = 0$ (or both), the bracket becomes the usual ring-theoretic commutator.

VOSAs contd.

This data satisfies the following axioms:

- Each vector has an real eigenvalue (known as the *weight*) under the operator $L(0)$ which puts the vector into a weight space V_r where

$$V_r = \{v \in V : L(0)v = rv\}$$

where r is a real number. We then have

$$V = \bigoplus_{r \in \mathbb{R}} V_r$$

with $\dim V_r < \infty$. We will focus on half-integral gradings in particular:

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n = \left(\bigoplus_{n \in \mathbb{Z}} V_{\bar{0},n} \right) \oplus \left(\bigoplus_{n \in \mathbb{Z}+\frac{1}{2}} V_{\bar{1},n} \right)$$

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- There exists a positive integer N such that:

$$(w-z)^N[Y(u,w), Y(v,z)] = 0$$

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$$= \sum_{m,n\in\mathbb{Z}} [u(m),v(n)]w^{-m-1}z^{-n-1}$$

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for all $u, v \in V$, N sufficiently large (locality). These operators are then said to be *local of order N*. We denote this by $Y(u,w) \overset{N}{\sim} Y(v,z)$.

Example - The Free Fermion VOSA

We consider the example of the free fermion model, which has a single generator ψ whose operators satisfy commutation relations

$$[\psi(m), \psi(n)] = \delta_{m+n+1,0}$$

Every vector v in this VOSA has a decomposition

$$v = \psi(-k_1) \cdots \psi(-k_m)\mathbf{1}$$

for $k_1 < k_2 < \cdots < k_m$, $k_i \in \mathbb{Z}$. We can then derive the weight for each vector by

$$wt(v) = \sum_{i=1}^{m} \left(k_i - \frac{1}{2} \right)$$

so $Y(\psi, z)$ generates the entire space. As a (rather trivial) example, we know $\psi = \psi(-1)\mathbf{1}$, so $wt(\psi) = \frac{1}{2}$.

Example - The Free Fermion VOSA

Compare this to the Heisenberg VOA where the generator commutator relations are

$$[h(m), h(n)] = m\delta_{m+n,0}$$

and every vector $v \in V$ can be written as

$$v = h(-1)^{k_1} h(-2)^{k_2} \cdots h(-r)^{k_r} \mathbf{1}$$

where k_i are non-negative integers. Then

$$wt(v) = 1 \cdot k_1 + 2 \cdot k_2 + \cdots + r \cdot k_r$$

Next time

Partition functions, Zhu recursion, twisted elliptic functions...

References

- Mason, G., Tuite, M.P., Zuevsky, A.: Torus n-point functions for R-graded vertex operator superalgebras and continuous fermion orbifolds. Commun. Math. Phys. **283**, 305-342 (2008)
- Kac, V.: Vertex Operator Algebras for Beginners. University Lecture Series, Vol. **10**, Providence, RI: Ameri. Math. Soc., 1998