Genus Two Zhu Theory for Fermionic VOSAs I

Mike Welby

27th October, 2017

Introduction

In this talk we will discuss the development of a genus two analogue of the Zhu recursion formula developed by Mason, Tuite and Zuevsky for a genus one vertex operator super-algebra (VOSA), or equivalently, a VOSA of the recursion formula found by Gilroy and Tuite.

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- A Virasoro vector $\omega \in V$

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- For all u, v in V, we have:

$$(z - w)^N [Y(u, z), Y(v, w)] = 0$$

where $[,]$ is the commutator defined by:

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- $Y(\mathbf{1}, z) = Id_V$
- $Y(u, z)\mathbf{1} = u + O(z)$

VOSAs continued

- $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-1}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

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- $Y(L(-1)v, z) = \frac{d}{dz} Y(v, z)$

Modular forms and Elliptic functions

We now define modular forms. A modular form is a function $f(\tau)$ on the upper-half complex plane $\mathbb{H}$ which:

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$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(z)$$

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  where $a, b, c, d, \in \mathbb{Z}$ and $ad-bc=1$, for some non-negative (even) integer k (called the *weight* of the form)
- has a Fourier expansion

$$f(\tau)=\sum_{n=0}^{\infty} a(n)q^n$$

  where $q = exp(2\pi i\tau)$. This converges for $|q| < 1$ (i.e. $\Im(\tau) > 0$)

Modular forms and Elliptic Functions

The examples of interest here are the Eisenstein series

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n=0}^{\infty}\sigma_{k-1}(n)q^n$$

where q is as before, B_k is a Bernoulli number and $\sigma_{k-1}(n)$ is the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

The E_k also have an alternative series representation:

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(n-1)!}\sum_{r\geq 0}\frac{r^{k-1}q^r}{1-q^r}$$

Following on from the E_k above we define:

$$P_n(z,\tau) = \frac{1}{z^n} + \sum_{k=2}^{\infty}\binom{k-1}{n-1}E_k(\tau)z^{k-n}$$

Note that there is no contribution from the odd k cases as then the E_k are trivial forms.

Twisted Functions

We can add additional parameters to these functions, which now become twisted Eisentein series and elliptic functions:

$$P_n \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \frac{1}{z^n} + (-1)^n \sum_{k=2}^{\infty} \binom{k-1}{n-1} E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) z^{k-n}$$

where

$$E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) = -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!} \sum_{r \geq 0}' \frac{(r+\lambda)^{k-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}}$$

$$+ \frac{(-1)^k}{(k-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{k-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}$$

where $\phi, \theta \in U(1)$, $\phi = \exp(2\pi i \lambda)$. Note that if we set $\theta, \phi = 1$ then $E_k \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\tau)$ collapses to the classical Eisenstein series (mostly).

n-point Functions for VOSAs

The n-point function for a VOSA V is defined by

$$Z_V^{(1)}(g; v_1, z_1; \ldots; v_n, z_n; \tau)$$

$$= \mathrm{STr}_V(gY(q_1^{L(0)}v_1, q_1) \cdots Y(q_n^{L(0)}v_n, q_n)q^{L(0)-c/24})$$

where $g \in Aut(V)$ and $STr_V(A) = Tr_{V_{\bar{0}}}(A) - Tr_{V_{\bar{1}}}(A)$ for an operator A. It can also be naturally defined for a VOSA module M.

Zhu Recursion for VOSAs

n-point functions undergo *Zhu recursion* and can be expressed in terms of $(n-1)$-point functions as follows:

$$
\begin{aligned}
&Z_V^{(1)}(g; v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\
&= \delta_{\phi,1}\delta_{\theta,1} STr_V(go(v)Y(v_1, q_1)\cdots Y(v_n, q_n)q^{L(0)-c/24}) \\
&\quad + \sum_{k=1}^{n}\sum_{m\geq 0} p(v, \boldsymbol{v_{k-1}}) \cdot P_{m+1}\begin{bmatrix}\theta \\ \phi\end{bmatrix}(z - z_k, \tau) \\
&\quad \times Z_V^{(1)}(g; v_1, z_1; \ldots; v[m]v_k, z_k; \ldots; v_n, z_n; \tau)
\end{aligned}
$$

where $gv = \theta^{-1}v$, $\phi = \exp(2\pi i wt(v))$ and $p(v, \boldsymbol{v_{k-1}}) = (-1)^{p(v)[p(v_1)+\cdots+p(v_{k-1})]}$ for $r > 1$.

Genus Two

The idea is to use a sewing scheme introduced by Yamada and expanded on by Mason and Tuite to develop a genus two version of the above.

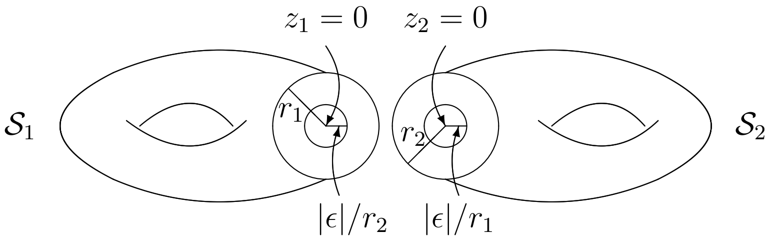

Fig. 1 Sewing Two Tori

More on this on the next talk.

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