Genus One Zhu Recursion for Vertex Operator Superalgebras

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Introduction

In this talk we will briefly outline genus one Zhu Recursion on vertex operator algebras (VOSA).

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- A Virasoro vector $\omega \in V$

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- For all u, v in V, we have:

$$(z - w)^N [Y(u, z), Y(v, w)] = 0$$

where $[,]$ is the commutator defined by:

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- $Y(\mathbf{1}, z) = Id_V$
- $Y(u,z)\mathbf{1} = u + O(z)$

VOSAs continued

- $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-1}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m,-n}c$$

where c is a constant known as the central charge.

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- $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$

Modular forms and Elliptic functions

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$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(z)$$

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  where $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$, for some non-negative integer k (called the weight of the form)
- has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$$

  where $q = exp(2\pi i\tau)$. This converges for $|q| < 1$ (i.e. $\Im(\tau) > 0$)

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  as the transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ is an action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$, and that group is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n$$

where q is as before, B_k is a Bernoulli number and $\sigma_{k-1}(n)$ is the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Elliptic Functions

The E_k also have an alternative series representation:

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{r \geq 1} \frac{r^{k-1} q^r}{1 - q^r}$$

Following on from the E_k above we define elliptic Weierstrass functions:

$$P_n(z, \tau) = \frac{1}{z^n} + \sum_{n \geq k} \binom{k-1}{n-1} E_k(\tau) z^{k-n}$$

Note that there is no contribution from the odd k cases as then the E_k are trivial forms.

Twisted Functions

We can add additional parameters to these functions, which now become twisted Eisentein series and elliptic functions:

$$P_n \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \frac{1}{z^n} + (-1)^n \sum_{k=2}^{\infty} \binom{k-1}{n-1} E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) z^{k-n}$$

where

$$E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) = -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!} \sum_{r \geq 0}{}' \frac{(r+\lambda)^{k-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}}$$

$$+ \frac{(-1)^k}{(k-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{k-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}$$

where $\phi, \theta \in U(1)$, $\phi = \exp(2\pi i \lambda)$. Note that if we set $\theta, \phi = 1$ then $E_k \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\tau)$ becomes the classical Eisenstein series E_k.

n-point Functions for VOAs

We now define an n-point function for a VOA by:

$$Z_V^{(1)}(v_1, z_1; \dots ; v_n, z_n; \tau)$$
$$= Tr\left(Y\left(q_1^{L(0)} v_1, q_1\right) \cdots Y\left(q_n^{L(0)} v_n, q_n\right) q^{L(0)-c/24}\right)$$

where $q_i = \exp(z_i) = \sum_{n\geq 0} \frac{z_i^n}{n!}$ is a formal series in z_i.

Zhu Recursion

Zhu developed a recursion formula relating genus one n-point functions to $(n-1)$-point functions:

$$Z_V^{(1)}(v, z; v_1, z_1; \ldots; v_n, z_n; \tau)$$
$$= Tr_V \left(o(v) Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24} \right)$$
$$+ \sum_{k=2}^{n} \sum_{j \geq 0} P_{1+j}(z - z_k, \tau) \cdot Z_V^{(1)}(v_1, z_1; \ldots; v[j] v_k, z_k; \ldots; v_n, z_n; \tau) \tag{1}$$

where $o(v) = v(wt - 1)$ and $v[j]$ is the coefficient of z^{-j-1} in $Y[v, z] = Y(q_z^{L(0)} v, q_z - 1)$ with $q_z = \exp(z)$.

The VOSA Version

The n-point function for a VOSA V is defined, then, by

$$Z_V^{(1)}(g; v_1, z_1; \dots; v_n, z_n; \tau)$$

$$= STr_V \left(gY \left(q_1^{L(0)} v_1, q_1 \right) \cdots Y \left(q_n^{L(0)} v_n, q_n \right) q^{L(0)-c/24} \right)$$

where $g \in Aut(V)$ and $STr_V(A) = Tr_{V_{\bar{0}}}(A) - Tr_{V_{\bar{1}}}(A)$ for an operator A.

Zhu Recursion for VOSAs

The recursion formula for a VOSA V is quite similar in structure to that of a VOA:

$$\begin{aligned}
&Z_V^{(1)}(g; v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\
&= \delta_{\phi,1}\delta_{\theta,1} STr_V(go(v)Y(v_1, q_1)\cdots Y(v_n, q_n)) \\
&+ \sum_{k=1}^{n}\sum_{m\geq 0} p(v, v_1 \ldots v_{k-1}) \cdot P_{m+1}\begin{bmatrix}\theta \\ \phi\end{bmatrix}(z - z_k, \tau)\cdot \\
&Z_V^{(1)}(g; v_1, z_1; \ldots; v[m]v_k, z_k; \ldots; v_n, z_n; \tau)
\end{aligned} \tag{2}$$

where $gv = \theta^{-1}v$, $\phi = \exp(2\pi i wt(v))$ and
$p(v, v_1 \ldots v_{k-1}) = (-1)^{p(v)[p(v_1)+\cdots+p(v_{k-1})]}$ for $k > 1$.
We note that for $v, v_i \in V_{\bar{0}}, g = 1$, equation (2) reduces to (1).

References

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