Introduction to Vertex Algebras

Mike Welby

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Formal series

A *formal series* is an infinite series in some indeterminate z where convergence is not considered, i.e.

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

where the $a(n)$ are elements are members of some ring or vector space. Here we consider the ring of endomorphisms of a given vector space V (set of linear maps from V to itself under addition and composition).

Commutator

Some operators or other such objects do not always commute under the "multiplication" of an arbitrary ring. We define the commutator:

$$[A, B] = AB - BA$$

where A, B are in some ring R. In this context they are linear operators and $R = \mathrm{End}(V)$.

Locality

Bearing this in mind, we note that two formal series may not necessarily commute. There may exist a positive integer N such that

$$(z-w)^N[a(z), b(w)] = 0$$

We say $a(z)$ and $b(w)$ are *local of order* N if this is the case. We denote this by

$$a(z) \sim_N b(z)$$

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- An endomorphism T, known as *translation*

This data must satisfy the following the following axioms for all $u, v \in V$:

- $$\left.\begin{aligned} T\mathbf{1} &= 0 \\ Y(\mathbf{1}, z)u &= u \\ Y(u, z)\mathbf{1} &= u + \mathcal{O}(z) \end{aligned}\right\} \text{ (vacuum)}$$

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- $Y(u, z) \sim_N Y(v, z)$, for some N (locality)

Vertex Operator Algebras

A *vertex operator algebra* (VOA) is a VA with some extra axioms:

- There exists a Virasoro vector ω such that

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

where L_n satisfies the Virasoro Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C$$

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- The translation operator $T = L_{-1}$

The Heisenberg VOA

A classical example of a VOA is the Heisenberg algebra of central charge 1. This VOA is generated by $Y(a, z)$ whose components satisfy

$$[a_m, a_n] = m\delta_{m,-n}$$

The $Y(a, z)$ vertex operators are local of order 2, i.e.

$$(z - w)^2[Y(a, z), Y(a, w)] = 0$$

Another example is the Virasoro VOA, whose elements satisfy the commutation relations discussed on the previous slide. In that case, the $Y(\omega, z)$ operators are local of order 4.

V. Kac, *Vertex Algebras for Beginners, Second Ed.*, Univ. Lect. Ser. **10**, AMS; 1998.

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