Static Equilibrium Equations of a Model Red Blood Cell

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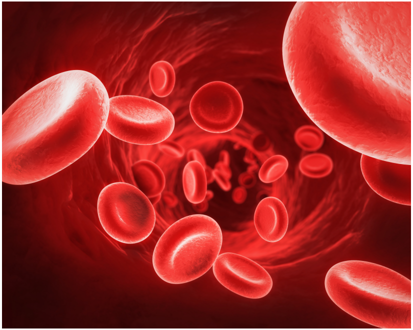
Introduction

- Mature red blood cell has a biconcave shape
- Adding water to plasma causes swelling
- Becomes ellipsoidal, eventually spherical
- Initial formation is reverse of this process
- Sphere buckles under excess of internal pressure



The Free Energy

Jenkins (1977): Only free energy function compatible with fluidity of surface and unaffected by rigid rotations depends at most on h and k.
Simplest form for free energy density of a surface is

$$w(h, k) = ch^2 + c_1 k \tag{1}$$

where h and k are mean and total curvatures of the surface.
Total energy is

$$W = \int_A w \, dA \tag{2}$$

The Free Energy

Considering variations in the energy and associated quantities, gives the normal equilibrium equation

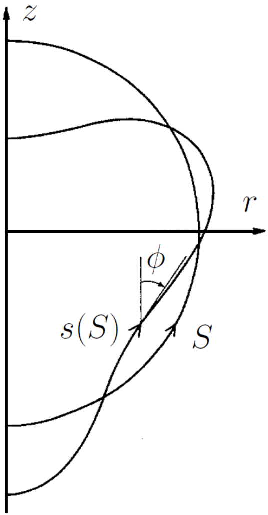
$$2h(\bar{d} + c(h^2 - k)) + c\partial_a(g^{ab}\partial_b h) = -\bar{p}, \tag{3}$$

where g^{ab} is the inverse of g_{ab}, the *first fundamental form*; d is a constant; and $\bar{p}$ is the difference between exterior and interior pressure.

Axisymmetric Deformations

Material parameters:

- Azimuthal angle θ
- Arc length S from axis of symmetry on sphere.
- Arc length on deformed surface is $s = s(S)$
- $\phi = \phi(S)$ is the angle between the tangent to a line $\theta =$ constant and the axis of symmetry.



Axisymmetric Deformations

$$\mathbf{X} = r(S)(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}) + z(S)\hat{\mathbf{k}} \tag{4}$$

Tangent vectors

$$\mathbf{e}_\theta = \partial_\theta \mathbf{X} = r(-\sin(\theta)\hat{\mathbf{i}} + \cos(\theta)\hat{\mathbf{j}}), \tag{5}$$

$$\mathbf{e}_S = \partial_S \mathbf{X} = \dot{s}\left(sin(\phi))\left(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}\right) + \cos(\phi)\hat{\mathbf{k}}\right), \tag{6}$$

with unit normal

$$\mathbf{n} = \frac{\mathbf{e}_\theta \times \mathbf{e}_S}{|\mathbf{e}_\theta \times \mathbf{e}_S|}$$
$$= \cos(\theta)\cos(\phi)\hat{\mathbf{i}} + \sin(\theta)\cos(\phi)\hat{\mathbf{j}} - \sin(\phi)\hat{\mathbf{k}}. \tag{7}$$

Calculating g_{ab} (with components $\mathbf{e}_a \cdot \mathbf{e}_b$) and the second fundamental form k_{ab} (with components $-\partial_a \mathbf{n} \cdot \mathbf{e}_b$) allows us to calculate the curvatures

$$
\begin{aligned}
k &= \det k^a_b = \det(k_{bc} g^{ca}) \\
&= \begin{vmatrix} \frac{-\cos\phi}{r} & 0 \\ 0 & \frac{r}{R}\dot{\phi} \end{vmatrix} \\
&= \dot{\phi}\frac{\cos\phi}{R}, \qquad (8) \\
h &= \frac{1}{2} k^a_a \\
&= \frac{1}{2}(k^1_1 + k^2_2) \\
&= \frac{1}{2}\left(\frac{r}{R}\dot{\phi} - \frac{\cos\phi}{r}\right). \qquad (9)
\end{aligned}
$$

Axisymmetric Deformations

Eliminating k, introducing transverse shear $q = \dot{h}\frac{r^2}{\sin^2(S)}$ gives

$$\dot{h} = \frac{\sin^2(S)}{r^2} q. \tag{10}$$

$$\dot{q} = -2h\left[d + h^2 + \frac{\cos\phi}{r}\left(2h + \frac{\cos(\phi)}{r}\right)\right] - p. \tag{11}$$

$$\dot{\phi} = \frac{2\sin(S)}{r} h + \frac{\sin(S)}{r^2}\cos(\phi), \tag{12}$$

$$\dot{r} = \frac{\sin(S)}{r}\sin(\phi), \tag{13}$$

$$\dot{z} = \frac{\sin(S)}{r}\cos(\phi), \tag{14}$$

(15)

$$q(0) = q(\pi) = 0, \phi(0) = \frac{\pi}{2}, \phi(\pi) = -\frac{\pi}{2}, r(0) = 0, z\left(\frac{\pi}{2}\right) = 0, \tag{16}$$

We expand $r \sim r_0(S) + \epsilon r_1(S)$ etc. to give

$$\ddot{h}_1 + \cot(S)\dot{h}_1 + ph_1 = 2d_1, \quad (17)$$

$$\dot{r}_1 = \phi_1 \sin(S) - r_1 \cot(S), \quad (18)$$

$$\dot{z}_1 = \phi_1 \cos(S) - r_1, \quad (19)$$

$$\dot{\phi}_1 = 2h_1 - \phi_1 \cot(S), \quad (20)$$

with

$$\dot{h}_1(0) = \dot{h}_1(\pi) = \phi_1(0) = \phi_1(\pi) = r_1(0) = z_1\left(\frac{\pi}{2}\right) = 0. \quad (21)$$

Solutions

Solutions for h_1 compatible with the boundary conditions are $h_1 = P_l(\cos(S))$, with $p = l(l+1)$, $l = 2, 3, \ldots$.
For $l = 2$ we have $p = 6$ so

$$
\begin{aligned}
h_1 &= \frac{1}{2}(3\cos^2(S) - 1), && (22)\\
\phi_1 &= \sin(S)\cos(S), && (23)\\
r_1 &= \frac{1}{4}\sin^3(S), && (24)\\
z_1 &= \frac{1}{16}\cos(S) + \frac{5}{48}\cos(3S); && (25)
\end{aligned}
$$

we add these to the known quantities h_0, ϕ_0, r_0, z_0 for the sphere, ...

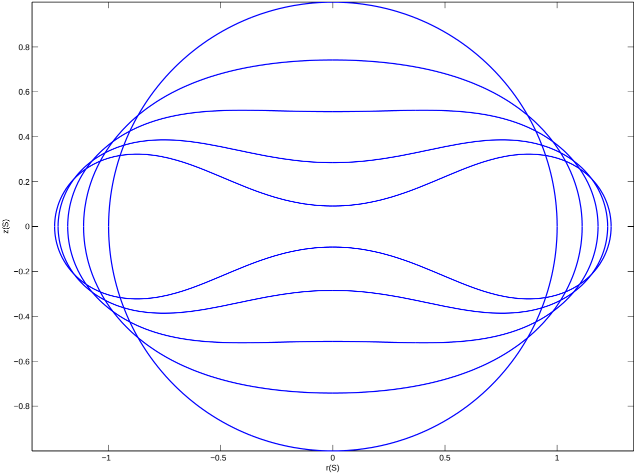


Figure: Cross sections of the axisymmetric deformation for $p = 6.0$ (spherical), $6.5, 7.0, 7.5, 7.9$; deformation increases with pressure

References

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