Self-Stress in Rigidity Theory

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Bar and joint Framework

Framework

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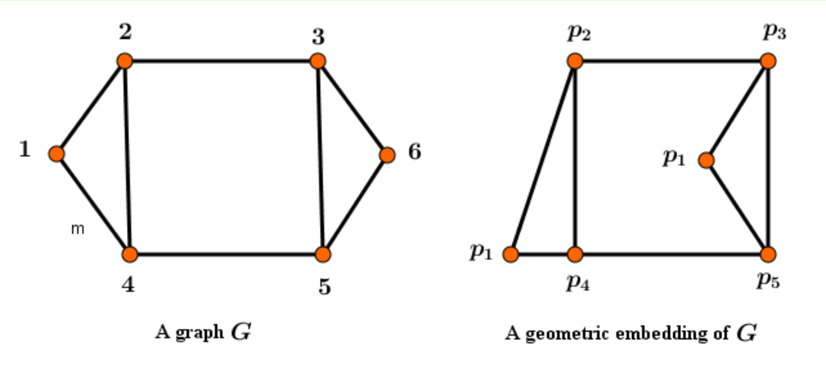

Infinitesmal rigidity

Given $\mathcal{F} = (G, p)$ with $|V| = n$ and $p = (p_1, ..., p_n)$. An infinitesimal flex $q = (q_1, ..., q_n) \in \mathbb{R}^{dn}$ is a vector satisfying $\langle p(i) - p(j) \rangle . \langle q(i) - q(j) \rangle = o$ for all edges $ij \in$.

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Rigidity Matrix

$$R_G(p) = \begin{pmatrix} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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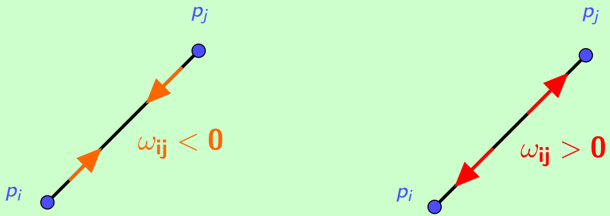

Self-Stress Definition

Self-stress

Let $\mathcal{F} = (G, p)$ be a framework. A stress ω of the framework $\mathcal{F}$ is called a self-stress if for each vertex $i \in V$ the following equilibrium condition is satisfied

$$\sum_{j:ij \in E} \omega_{ij}(p_i - p_j) = 0$$

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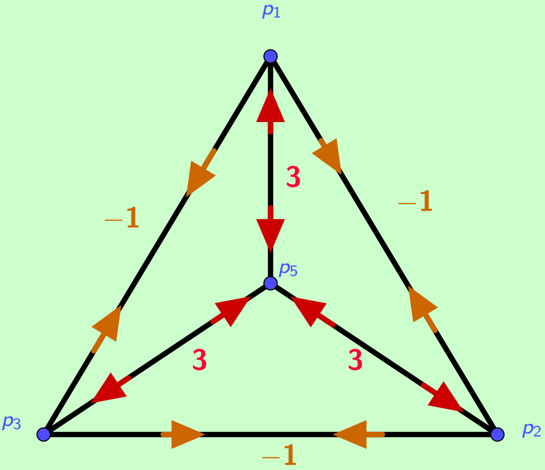
Zero Self-Stress

A stress ω is called trivial or zero self-stress if $\omega_i j = 0$ for all $ij \in E$.

An Example of Self-Stress

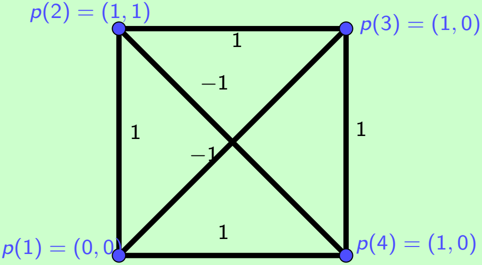


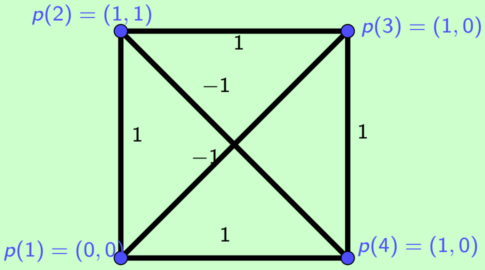
Stress matrix

Stress matrix

Let $\mathcal{F} = (G, p)$ be a framework and $\omega = (..., \omega_{ij}, ...)$ be a self-stress of $\mathcal{F}$. The stress matrix of $\mathcal{F}$ associated with ω is a symmetric matrix of size $|V| \times |V|$ with rows and columns indexed by vertices in V such that

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & ij \in E \\ \sum\limits_{k \in V : ik \in E} \omega_{ik} & i = j \\ o & Otherwise \end{cases}$$


$p(2) = (1, 1)$
$p(3) = (1, 0)$
1
−1
1
−1
1
1
$p(1) = (0, 0)$
$p(4) = (1, 0)$



$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Self Stress and Rigidity Matrix

Observation

A stress ω is a self-stress if and only if in the left null space of the rigidity matrix, i.e. $\omega R_G(p) = 0$.

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Independent framework

A framework $\mathcal{F} = (G, p)$ is called Independent framework if the rigidity matrix $R_G(p)$ has independent rows. Equivalently,there is only the zero self-stress.

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Isostatic framework

A framework $\mathcal{F} = (G, p)$ is called an isostatic if it is infinitesimal rigid and independent.

References

1. W. Whiteley. Vertex splitting in isostatic frameworks. Struc. Top., 16:23 -30, 1989.
2. A. Y. Alfakih. On bar frameworks, stress matrices and semidefinite programming. Math. Program., 129(1, Ser. B):113–128, 2011.
3. S. J. Gortler, A. D. Healy, and D. P. Thurston. Characterizing generic global rigidity. Amer. J. Math., 132(4):897–939, 2010.