Wrinkles in the opening angle method

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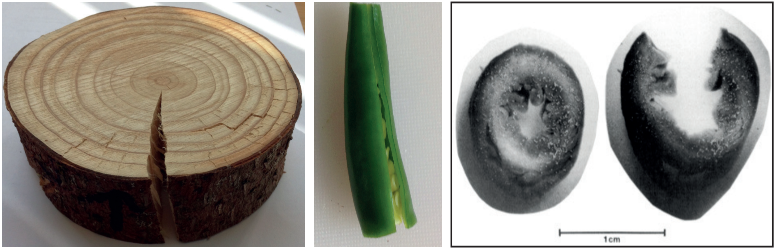
Residual stresses

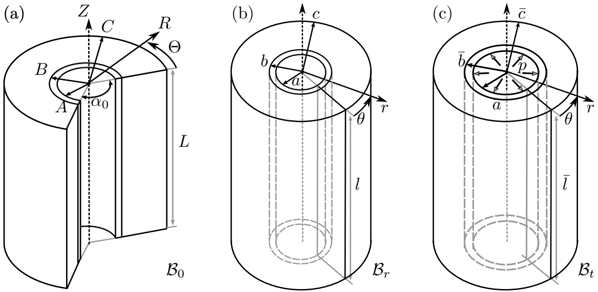
- Many biological structures experience a state of stress even if no external load is being applied. This is called **residual stress**.
- Can be demonstrated by cutting a cylindrical section of the structure radially. It will open up, revealing that they were under circumferential residual stresses



Left: slice of an Irish Ash tree; Middle: a green chilli pepper; Right: Equatorial slice of rat heart (taken from [1]).

Opening angle method

- When the cylindrical section is cut open, it goes from a state of residual stress to a state of (approximately) zero stress.
- The **opening angle** of the sector gives a measure of the residual stress.
- Residual stresses can then be modelled by running a **backwards simulation** of this scenario.



Two-layer model

We let $\{R, \Theta, Z\}$ and $\{r, \theta, z\}$ denote the cylindrical coordinate systems in which the geometries of the (undeformed) circular sector and the closed tube, respectively, are delimited.

The circular sector of opening angle α_0 consists of a **stiff inner layer** (coating) ($A \leq R \leq B$) and a **softer outer layer** (substrate) ($B \leq R \leq C$).

The deformation is then described by the mapping

$$r = r(R), \qquad \theta = k\Theta, \qquad z = \lambda_z Z, \tag{1}$$

where

$$k = \frac{2\pi}{2\pi - \alpha_0} > 1 \tag{2}$$

is a measure of the opening angle and $\lambda_z \geq 1$ is the uniform axial stretch. We let $a = r(A)$, $b = r(B)$, $c = r(C)$.

Two-layer model

In the absence of body forces, the only non-trivial **equation of equilibrium** is

$$\frac{\partial \sigma_{rr}}{\partial r}^{(l)} + \frac{\sigma_{rr}^{(l)} - \sigma_{\theta\theta}^{(l)}}{r} = 0 \quad (l = s, c), \tag{3}$$

where the radial and circumferential stresses are given by

$$\sigma_{rr}^{(l)} = -q^{(l)} + \lambda_1 \frac{\partial W^{(l)}}{\partial \lambda_1}, \quad \sigma_{\theta\theta}^{(l)} = -q^{(l)} + \lambda_2 \frac{\partial W^{(l)}}{\partial \lambda_2}, \tag{4}$$

respectively, where $q^{(l)}$ is an arbitrary scalar, the λ_i are principal stretches. Here s and c refer to the the outer layer and the inner layer, respectively.

For the boundary conditions we prescribe an internal pressure P, along with perfect bonding at the interface and a traction free outer face:

$$\sigma_{rr}^{(c)}(a) = -P, \qquad \sigma_{rr}^{(s)}(b) = \sigma_{rr}^{(c)}(b), \qquad \sigma_{rr}^{(s)}(c) = 0. \tag{5}$$

Two-layer model

For convenience, we introduce the quantities,

$$x \equiv k\lambda_z \frac{r^2}{R^2}, \qquad x_a \equiv k\lambda_z \frac{a^2}{A^2}, \qquad x_b \equiv k\lambda_z \frac{b^2}{B^2}, \qquad x_c \equiv k\lambda_z \frac{c^2}{C^2}, \quad (6)$$

This allows us to transform the strain energy densities $W^{(l)}(\lambda_1, \lambda_2, \lambda_3)$ into a function of a single variable, $\hat{W}^{(l)}(x)$. Making this transformation, integrating the equilibrium equation and applying the boundary conditions, we find that the inflating pressure P is

$$P = \int_{x_a}^{x_b} \frac{\widehat{W}^{(c)}_{,x}(x)}{1-x} \mathrm{d}x + \int_{x_b}^{x_c} \frac{\widehat{W}^{(s)}_{,x}(x)}{1-x} \mathrm{d}x \quad (l = s, c). \quad (7)$$

For a given axial stretch λ_z, internal pressure P and strain energy densities $\hat{W}^{(l)}(x)$, the new geometry can be determined by solving the above equation along with

$$x_b(\epsilon_B + 1) = \epsilon_B + x_a, \qquad x_c(\epsilon_C + 1) = \epsilon_C + x_a. \quad (8)$$

where $\epsilon_B = B^2/A^2 - 1$ and $\epsilon_C = C^2/A^2 - 1$.

Instability

However, the previously discussed deformation may not be stable. Here we study the stability of a coated sector closed into a pressurized cylinder. We signal the onset of instability by the existence of **small-amplitude wrinkles**, solutions to the incremental equations of equilibrium. We seek **decaying sinusoidal solutions**.

It can be shown that the wrinkles exist when the following boundary value problem is solved for $\mathbf{z}^{(l)} = \mathbf{z}^{(l)}(x)$, $(l = s, c)$, the 2×2 Hermitian *surface impedance matrix* [3].

(i) Initial condition: $\mathbf{z}^{(c)}(x_a) = \mathbf{0}$;

(ii) Numerical integration of the differential matrix Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{z}^{(l)} = \frac{1}{2x(1-x)}\left[\mathbf{z}^{(l)}\mathbf{G}_2^{(l)}\mathbf{z}^{(l)} + \mathrm{i}\left(\mathbf{G}_1^{(l)}\right)^{\dagger}\mathbf{z}^{(l)} - \mathrm{i}\mathbf{z}^{(l)}\mathbf{G}_1^{(l)} + \mathbf{G}_3^{(l)}\right], \tag{9}$$

in the coating $(l = c)$, from x_a to x_b;

(iii) Interfacial condition: $\mathbf{z}^{(c)}(x_b) = \mathbf{z}^{(s)}(x_b)$;

(iv) Numerical integration of the differential Riccati matrix equation (9) in the substrate ($l = s$), from x_b to x_c; and

(v) Target condition: $\det \mathbf{z}^{(s)}(x_c) = 0$.

In Eq.(9), † denotes the Hermitian transpose and the Stroh sub-matrices have components [2],

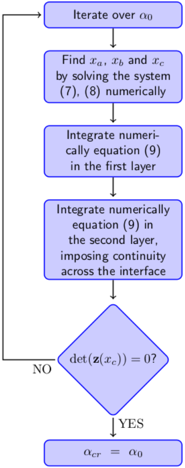
$$\mathbf{G}_1 = \begin{bmatrix} \mathrm{i} & -n \\ -n(1-\sigma) & -\mathrm{i}(1-\sigma) \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1/\alpha \end{bmatrix}, \quad \mathbf{G}_3 = \begin{bmatrix} \kappa_{11} & \mathrm{i}\kappa_{12} \\ -\mathrm{i}\kappa_{12} & \mathrm{i}\kappa_{22} \end{bmatrix}, \tag{10}$$

where the superscript "(l)" is understood, n denotes the wrinkling mode (number of wrinkles), and

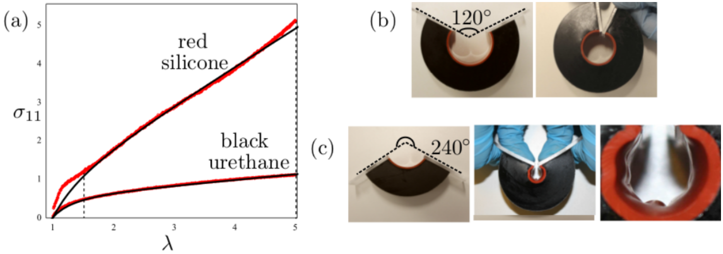
$$\begin{aligned} \kappa_{11} &= 2\beta + 2\alpha(1-\sigma) + n^2[\gamma - \alpha(1-\sigma)^2], \\ \kappa_{12} &= n(2\beta + \gamma + \alpha(1-\sigma^2), \\ \kappa_{22} &= \gamma - \alpha(1-\sigma)^2 + 2n^2(\beta + \alpha(1-\sigma). \end{aligned} \tag{11}$$

Here, in general,

$$\alpha = \frac{2x\widehat{W}_{,x}(x)}{k^2x^2-1}, \quad \gamma = k^2x^2\alpha, \quad \beta = 2x^2\widehat{W}_{,xx}(x) + x\widehat{W}_{,x}(x) - \alpha, \quad \sigma = \sigma_{rr}/\alpha. \tag{12}$$



For given reference geometry and material parameters, we can find the critical opening angle, $\alpha_{\mathrm{cr}} = \alpha_0$, at which wrinkles form when the sector is closed into an intact tube, i.e., the value of α_0 when the target condition is reached. Essentially, we implement the steps *(i)*-*(iv)* and iterate over α_0 until the target condition *(v)* is reached.

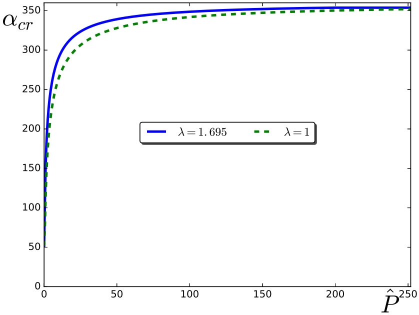


(a) Tensile tests for red silicone and black urethane. Curve-fitting to the Mooney-Rivlin models.
(b), (c) Sectors made of black urethane substrate and red silicone coating, with opening angles 120° and 240°. No wrinkles form in the former case, while six form in the latter case.

Implementing the algorithm for the dimensions and material parameters of this sector, we find that the critical opening angle is 201° (with $n = 4$).

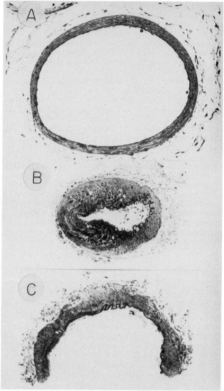
Experimental and numerical results

Next we perform the stability analysis using the dimensions and material parameters of a rabbit carotid artery, for varying pressure. We find that the critical opening angle for $P = 0$ is $\alpha_{\text{cr}} = 52^\circ$, significantly less than the recorded opening angle for the rabbit artery, 160°. We also find that the critical opening angle increases rapidly as the pressure increases.



Experimental and numerical results

Thus, our theory predicts that, *(i)* buckling should occur in the artery in the zero pressure state, and *(ii)* this buckling can be eliminated quickly by applying an internal pressure. This is in line with experiments on a rat's pulmonary artery.



Rat pulmonary artery at three different states: (A) Intact with low internal pressure of 15 mmHg and smooth internal surface; (B) Intact with no pressure and buckled internal face; (C) Cut open (taken from [5]).

References

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