Solids with a linear response in simple shear and torsion

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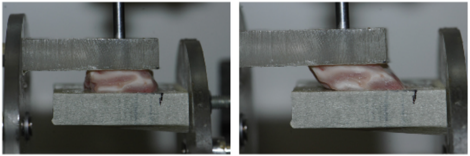
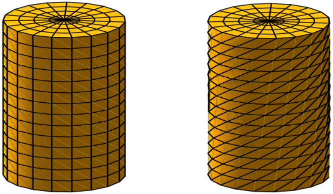
Simple shear and torsion

- Many soft materials have a linear stress response in simple shear and in torsion, including rubbers and soft tissues.





Simple shear and torsion

It can be shown that the Cauchy shear stress component T_{12} for simple shear is

$$T_{12} = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) K, \tag{1}$$

where K is the amount of shear, W is the strain energy and I_1, I_2 are principal invariants, and that the torque M for torsion of a cylinder is

$$M = 4\pi\psi \int_0^a r^3 \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) dr, \tag{2}$$

where ψ is the amount of twist, r is the radial distance and a is the radius of the cylinder.

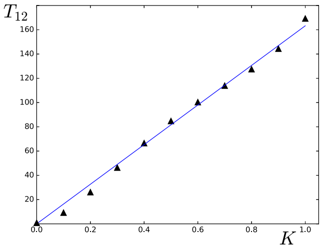
The Mooney-Rivlin model

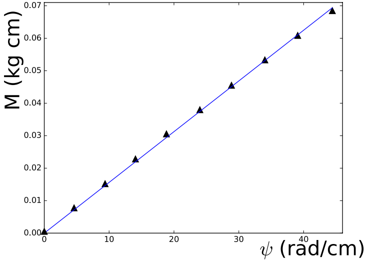
$$W_{\mathrm{MR}} = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2(I_2 - 3), \tag{3}$$

predicts a linear response in simple shear with

$$T_{12} = (C_1 + C_2)K, \tag{4}$$

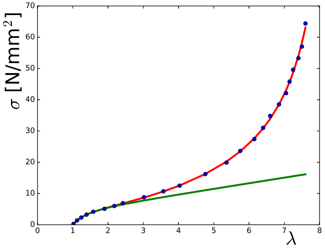
where C_1, C_2 are material parameters.





Mooney-Rivlin model

However, the Mooney-Rivlin model is not capable of capturing more complex behaviours like the upturn which occurs for the extension of a solid (strain-stiffening effect).



Derivation

In order to derive a strain energy W which leads to the desired linear response, we require

$$\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} = \text{const} = \frac{\mu}{2}. \tag{5}$$

where μ is the shear modulus.

Note that the Mooney-Rivlin model is a particular solution of this equation with $C_1 + C_2 = \mu$.

Thus the general solution may be written as

$$W = W_{\mathrm{MR}} + H(I_1, I_2). \tag{6}$$

Then, after substitution, we obtain a homogeneous partial differential equation for H,

$$\frac{\partial H}{\partial I_1} + \frac{\partial H}{\partial I_2} = 0. \tag{7}$$

The general solution of this equation is simply $H = H(I_1 - I_2)$. We call the corresponding class of solids, the *generalized Mooney-Rivlin materials*,

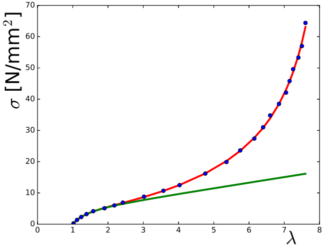
$$W = \tfrac{1}{2}C_1(I_1 - 3) + \tfrac{1}{2}C_2(I_2 - 3) + H(I_1 - I_2). \tag{8}$$

As an illustration we consider the following example of a generalized Mooney-Rivlin material,

$$W_{\mathrm{gMR}} = \tfrac{1}{2}C_1(I_1 - 3) + \tfrac{1}{2}C_2(I_2 - 3) - \tfrac{1}{2}C_3 J_m \ln\left(1 - \frac{I_1 - I_2}{J_m}\right). \tag{9}$$

Fitting

This model provides an excellent fit to the experimental data for extension of rubber.



Drawbacks

- The particular model mentioned involves four parameters.
- It can be shown that the entire class of generalized Mooney-Rivlin materials is not capable of modelling non-linear shear wave propagation.

References

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