Coupling normal forms via linear connection

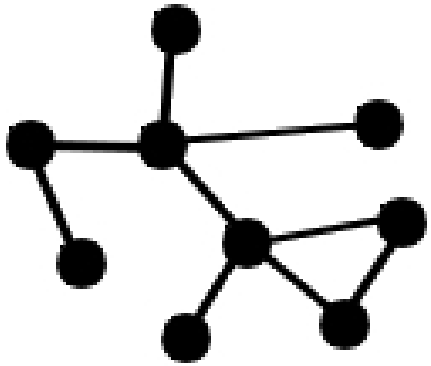
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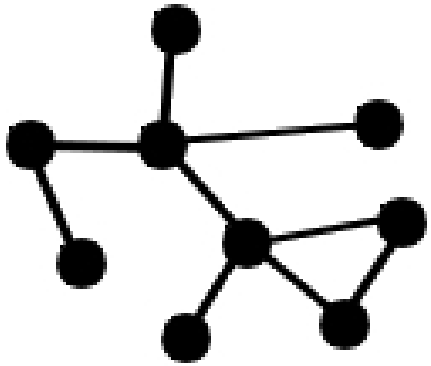
October 19, 2018

Dynamical Network:



Consider N identical dynamical systems (nodes)

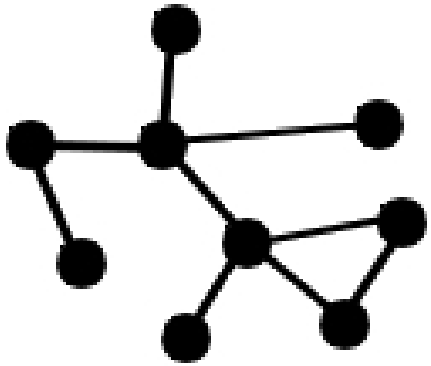
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$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} g(x_j, x_i), \quad \text{for} \quad i = 1 \ldots N \tag{1}$$

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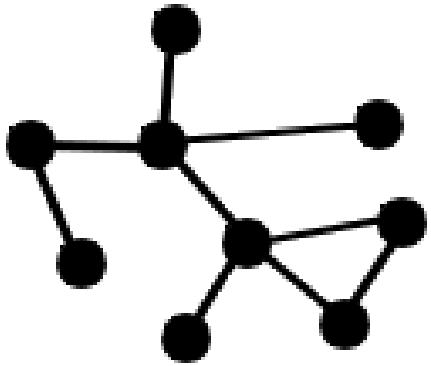


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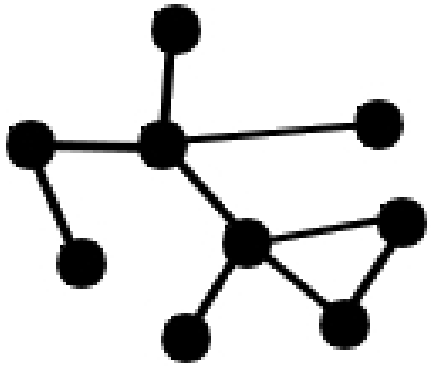


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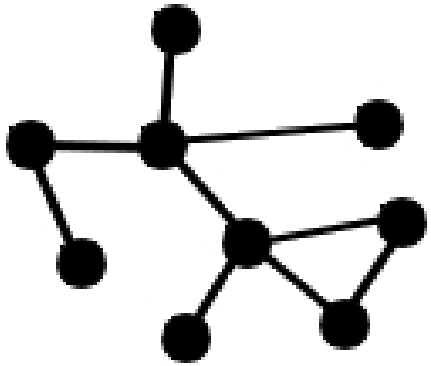


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- σ is the *coupling strength*.

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- The network is undirected $a_{ij} = a_{ji}$,
- The inner dynamics we consider are *saddle-node*, *transcritical*, and *pitchfork* bifurcation normal forms.

Saddle-node normal form

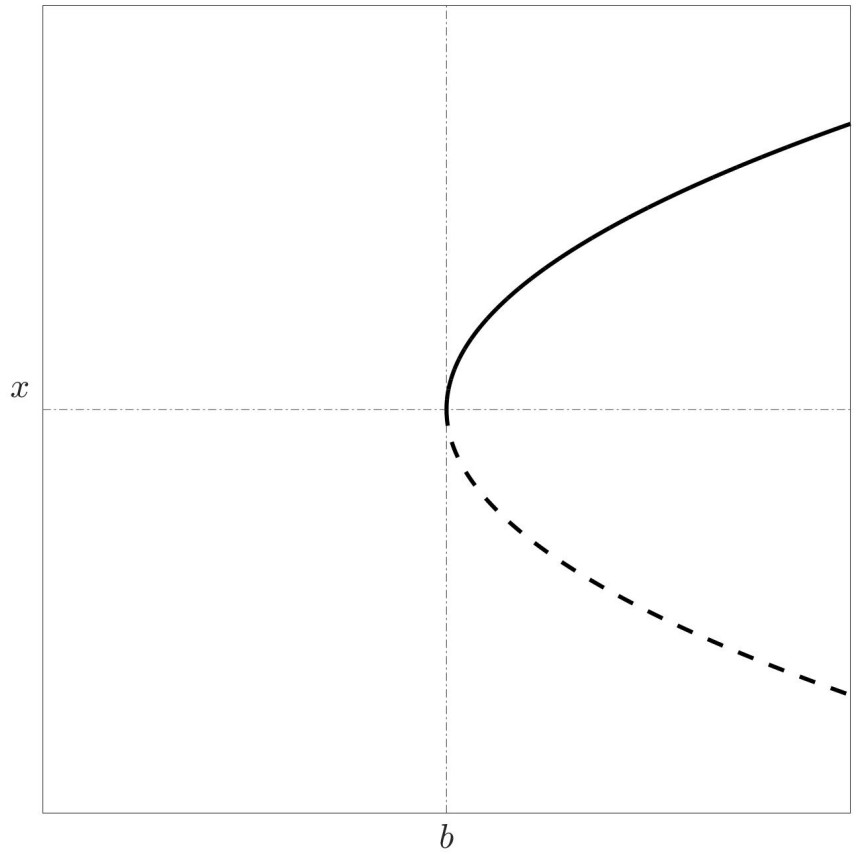
- $\dot{x} = b - x^2$

Saddle-node normal form

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Transcritical normal form

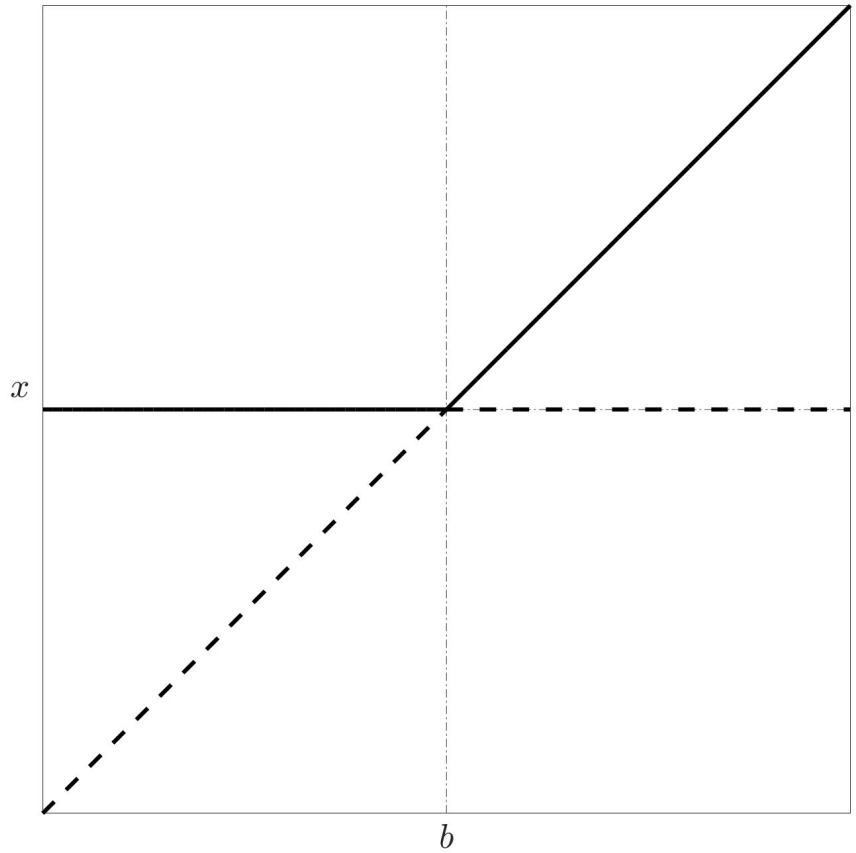
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Transcritical normal form

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Supercritical pitchfork normal form

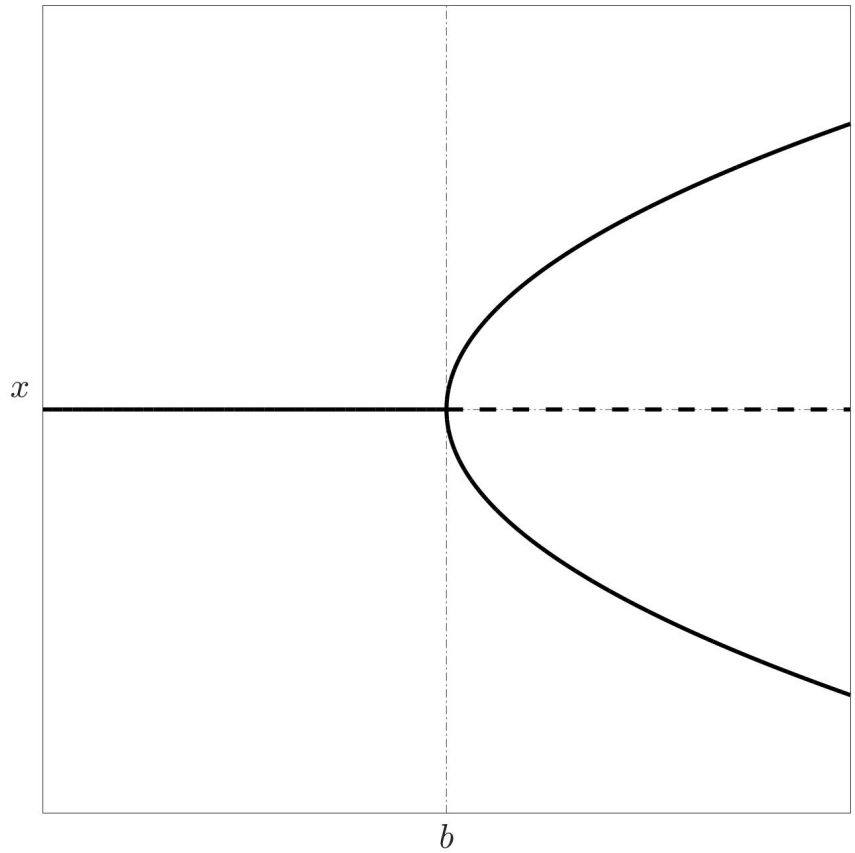
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Supercritical pitchfork normal form

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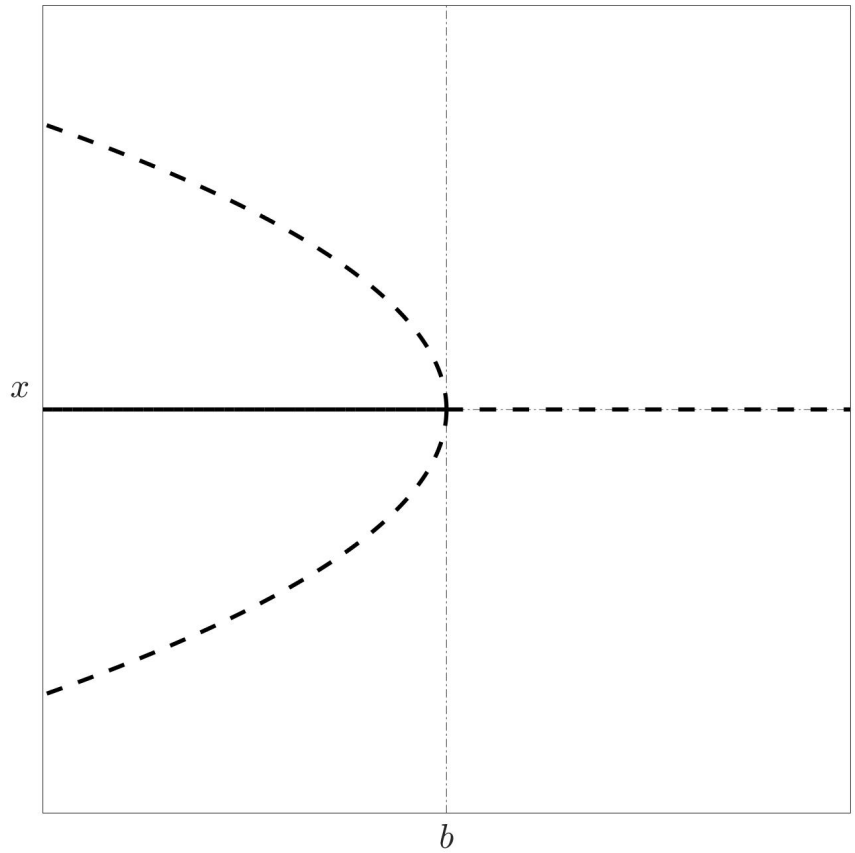
- $\dot{x} = bx + x^3$

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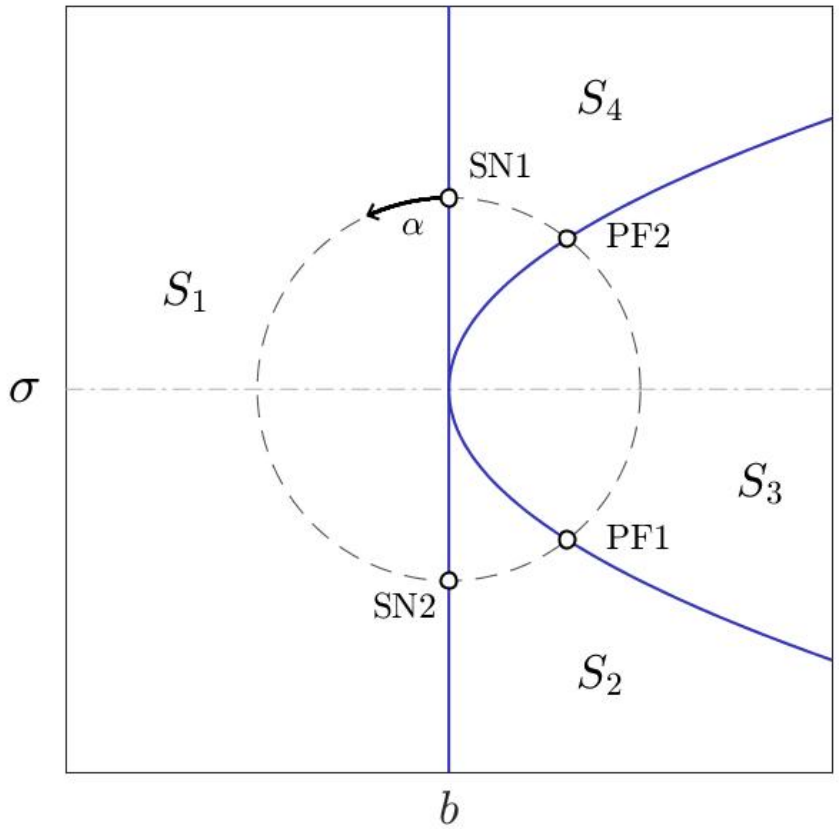


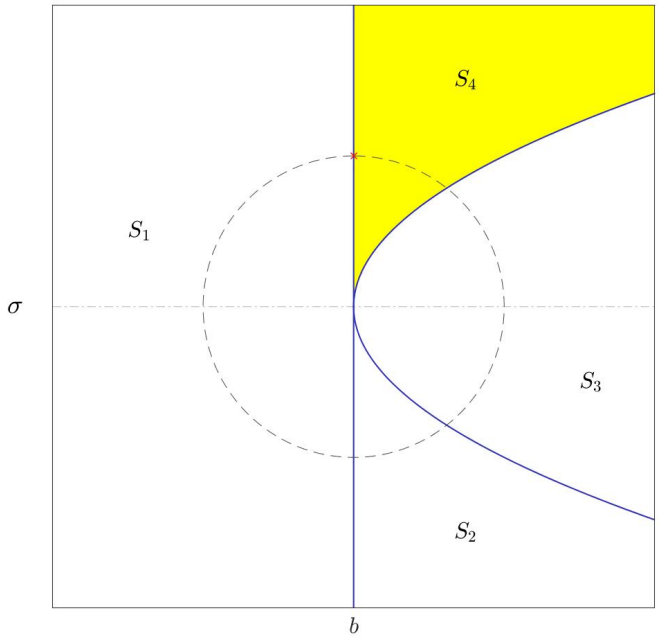
Coupling two saddle-node normal forms

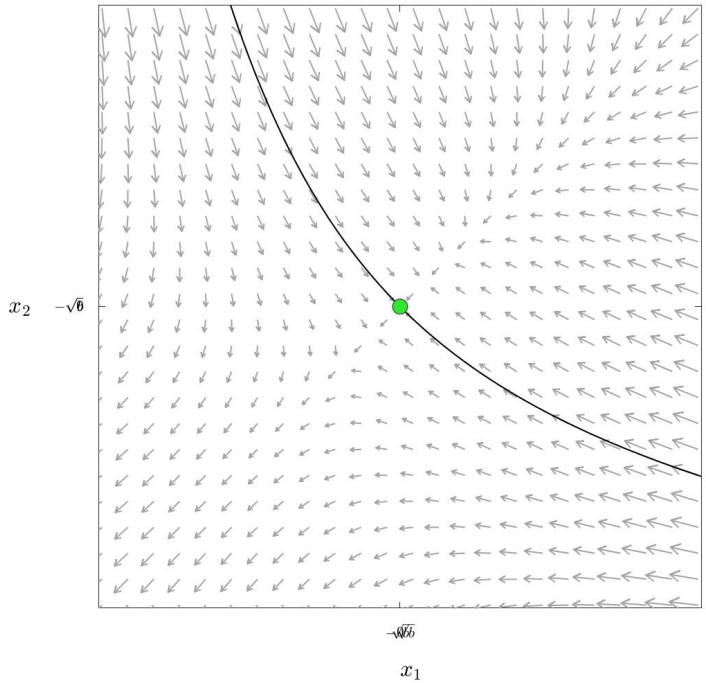
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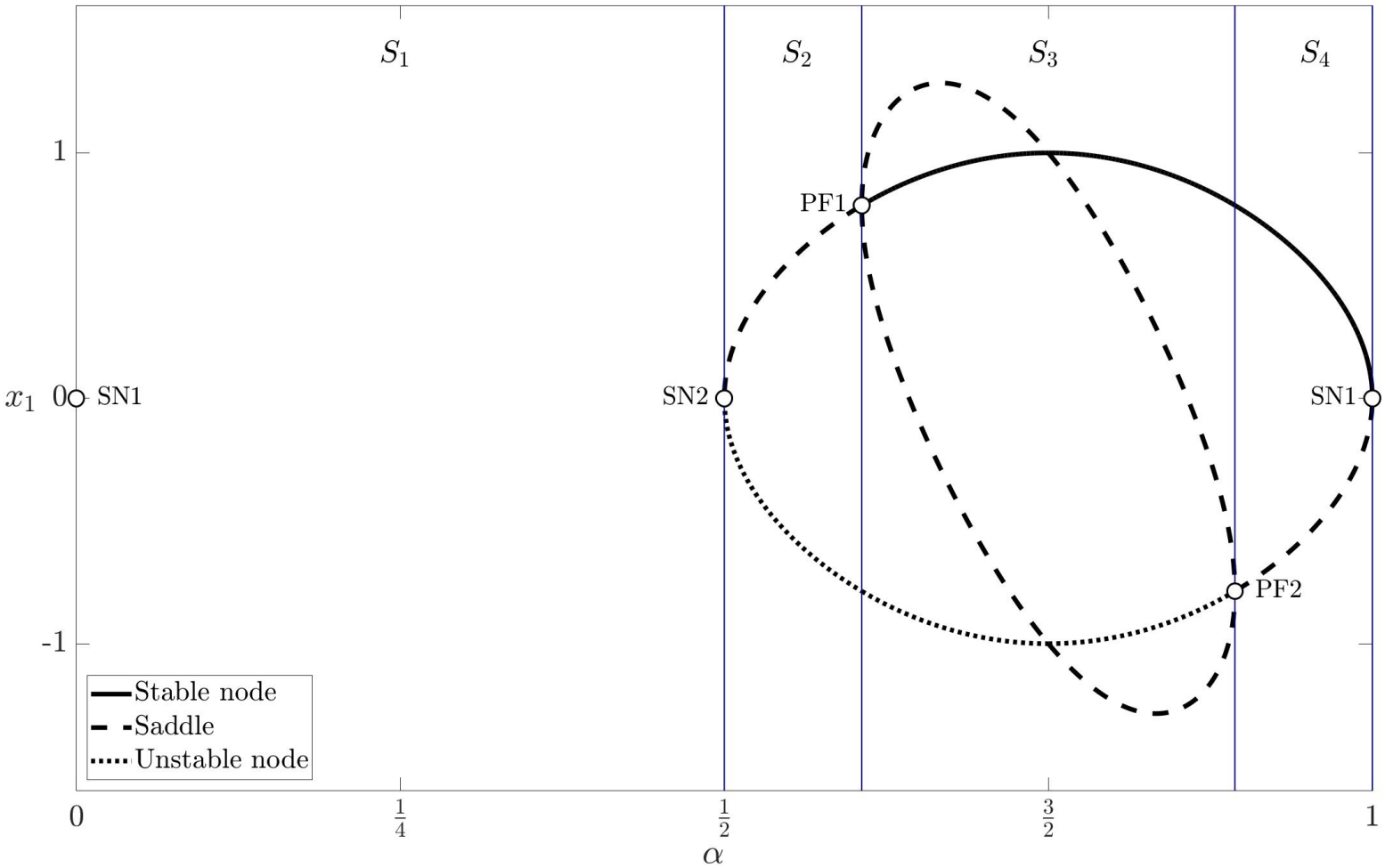
$$\begin{aligned} \dot{x}_1 &= b - x_1^2 + \sigma(x_2 - x_1), \\ \dot{x}_2 &= b - x_2^2 + \sigma(x_1 - x_2), \end{aligned} \tag{2}$$




S_4
S_1
σ
S_3
S_2
b


x_2
$-\sqrt{b}$
$-\sqrt[3]{b}$
x_1

Bifurcation diagram

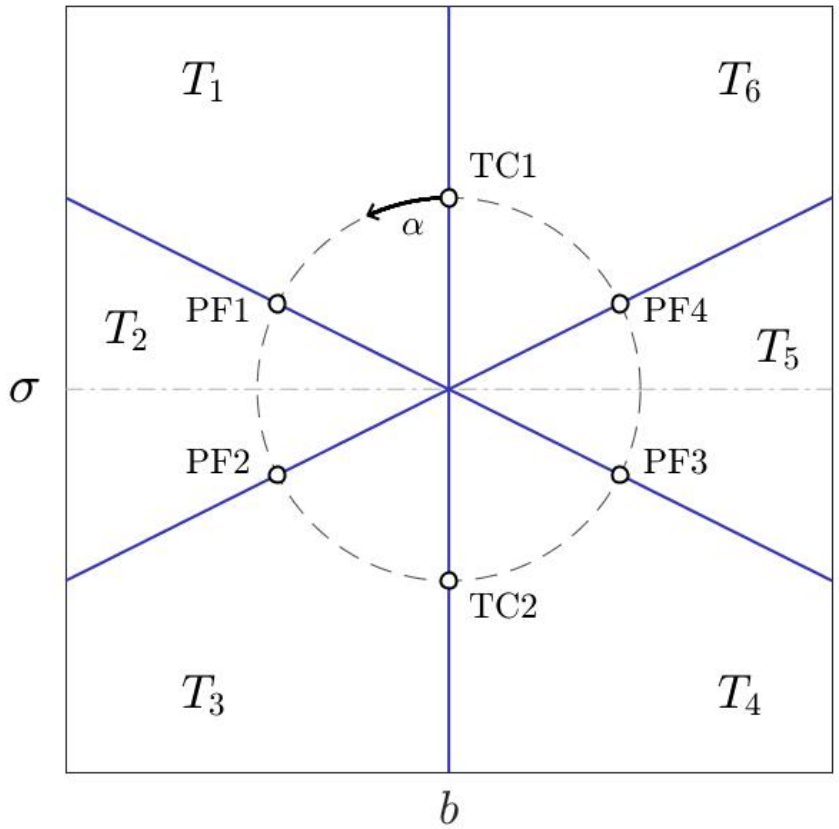


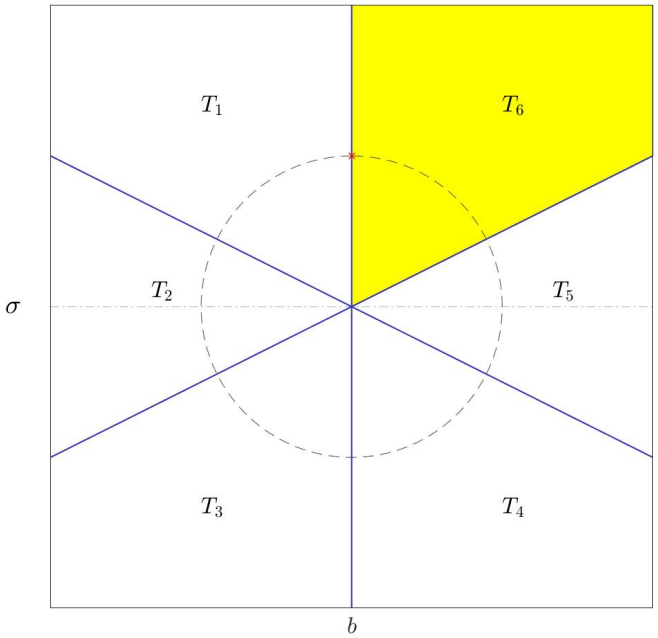
Coupling two transcritical normal forms

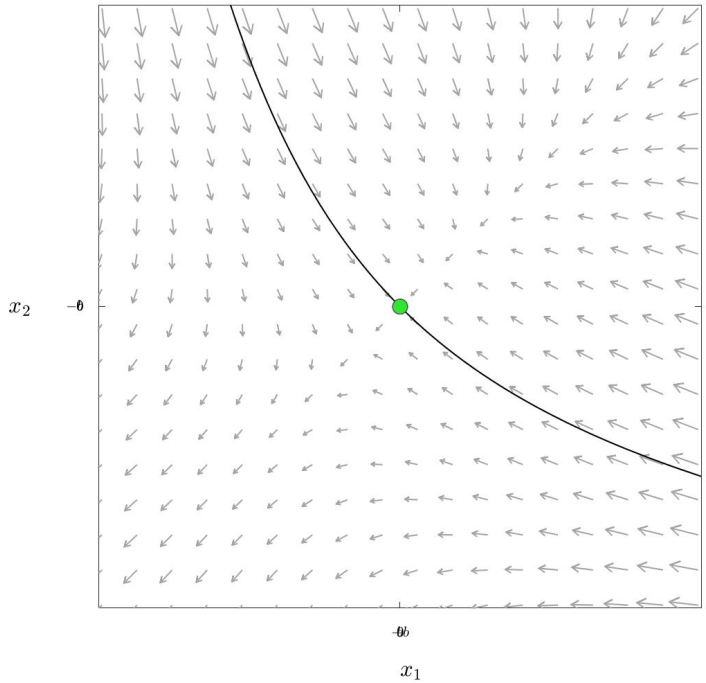
$$\begin{aligned} \dot{x}_1 &= bx_1 - x_1^2 + \sigma(x_2 - x_1), \\ \dot{x}_2 &= bx_2 - x_2^2 + \sigma(x_1 - x_2), \end{aligned} \tag{3}$$

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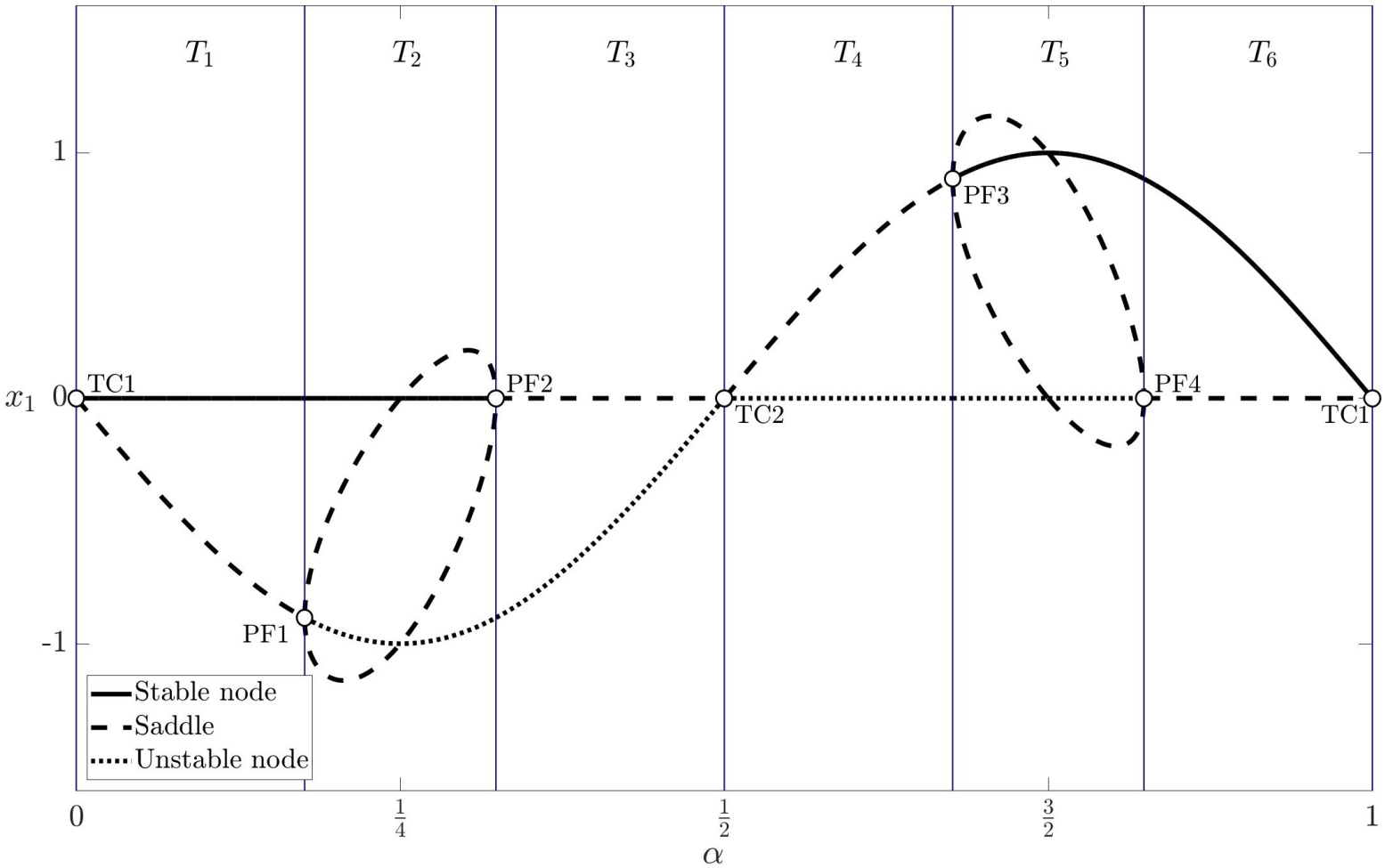
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T_1
T_6
T_2
T_5
T_3
T_4
σ
b


x_2
$-\theta$
$-\theta$
x_1

Bifurcation diagram

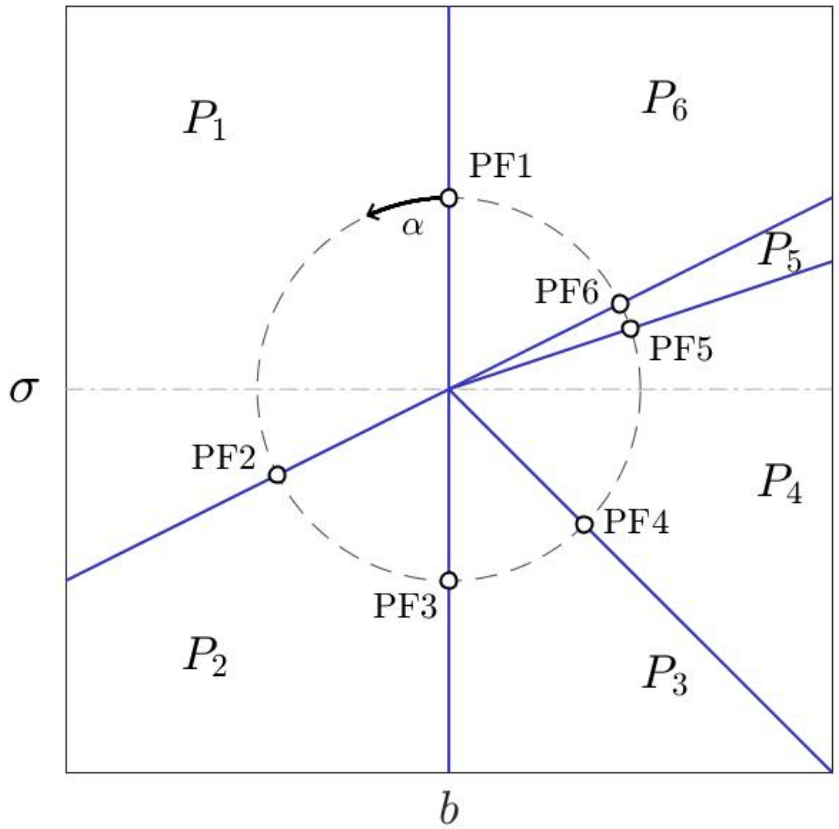


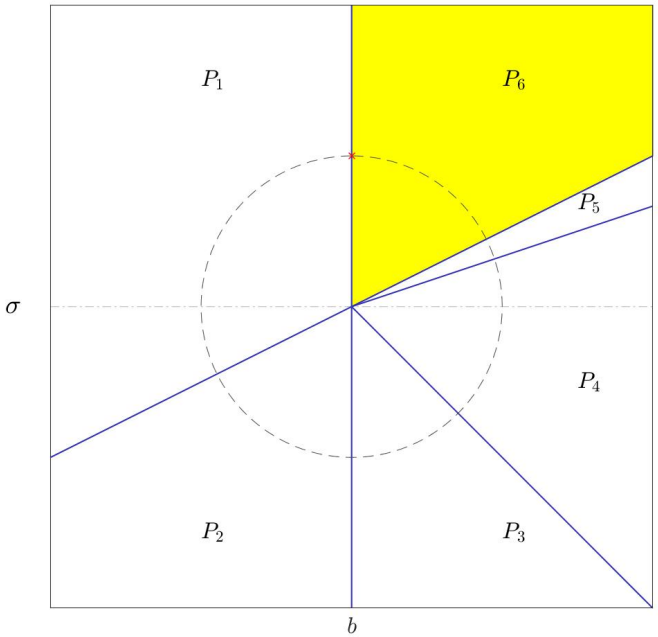
Coupling two pitchfork normal forms

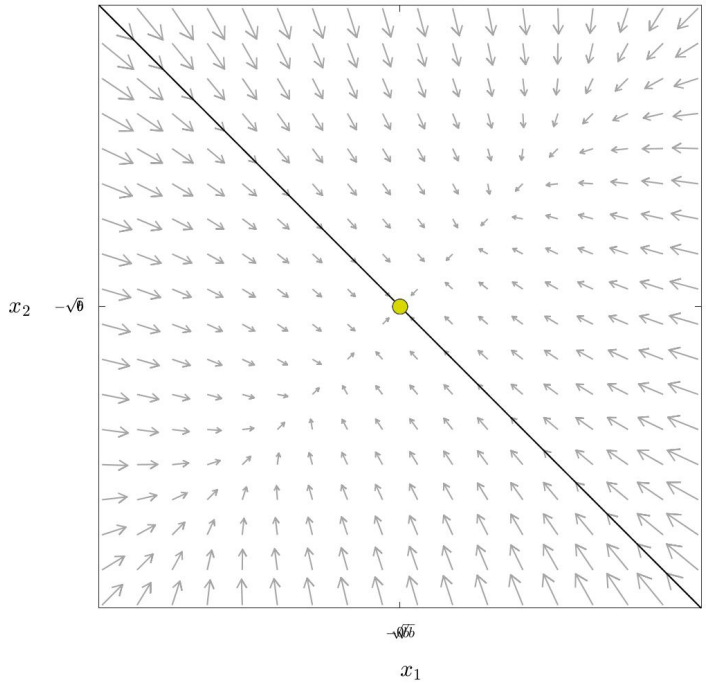
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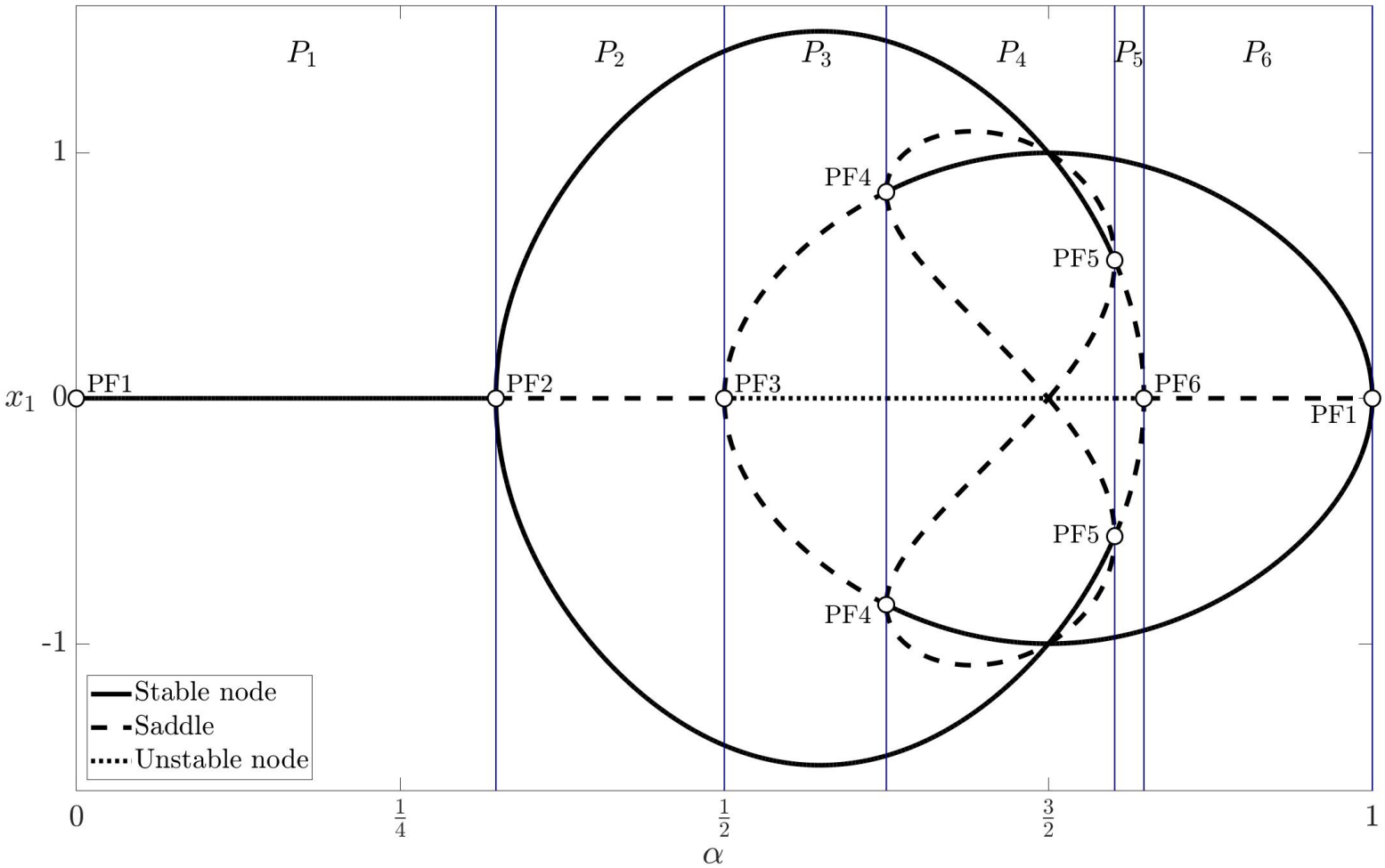
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P_1
P_6
P_5
P_4
P_2
P_3
σ
b


x_2
$-\sqrt{b}$
$-\sqrt{\sqrt{b}}$
x_1

Bifurcation diagram



Further research

- Higher dimensions

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij}(x_j - x_i) \qquad \text{for } i = 1, \ldots, N, \quad N > 2 \qquad (5)$$

$f(x_i) : \{\text{saddlenode, transcritical, pitchfork}\}$.

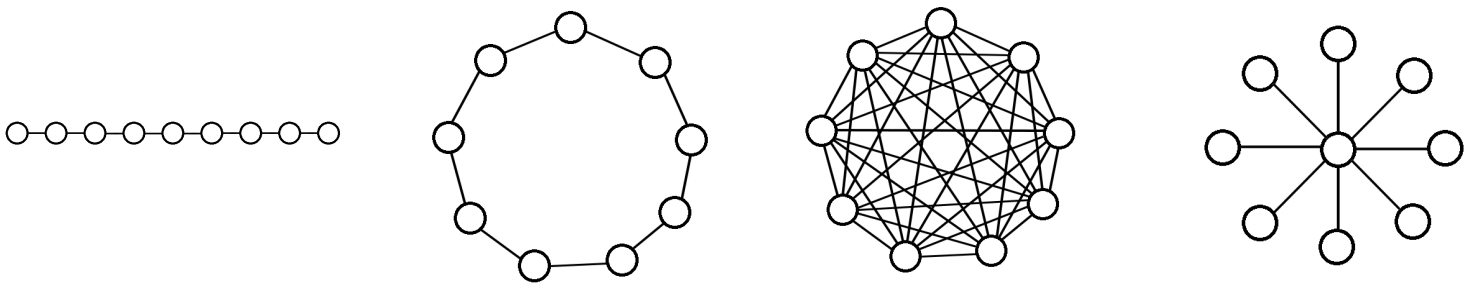
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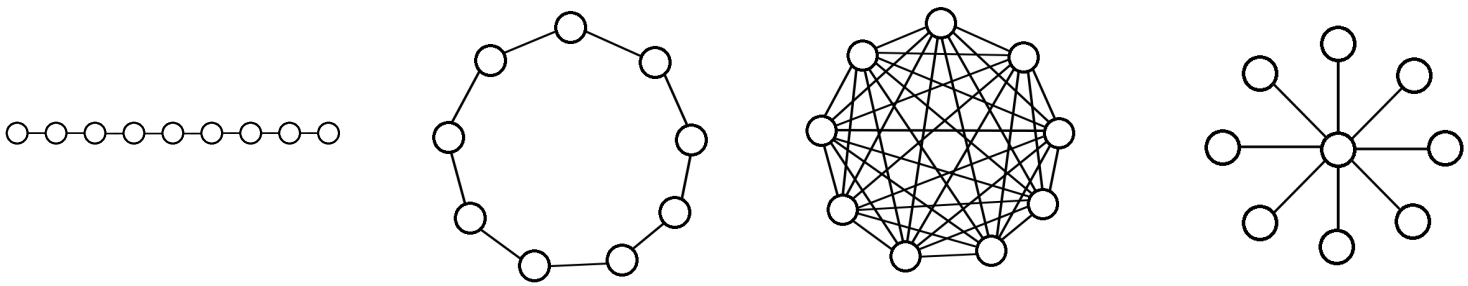
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- Detection region of simple dynamics

References



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