Mesh generation using a balanced norm for singularly perturbed reaction-diffusion problems

Róisín Hill

Supervisor: Dr Niall Madden

23 February, 2018

NUI Galway
OÉ Gaillimh

IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn

FEniCS PROJECT

Singularly perturbed reaction-diffusion problem

A one-dimensional singularly perturbed reaction-diffusion equation is

$$-\varepsilon^2 u''(x) + b(x)u(x) = f(x) \quad \text{on } \Omega = (0, 1),$$

with the boundary conditions

$$u(0) = u(1) = 0.$$

It is *singularly perturbed* in the sense that the positive real parameter ε may be arbitrarily small, but, if we formally set $\varepsilon = 0$, then the problem is ill-posed.

Error measurement

- Maximum norm, pointwise or global
  - $||e_N||_{\infty(0,1)} = \max|e_i|$;
- L^2 norm
  - $||e_N||_{L^2(0,1)} = \sqrt{\int_0^1 (e_N(x))^2 dx}$; and
- Energy norm
  - $||e_N||_{e(0,1)} = \sqrt{\varepsilon^2 \int_0^1 (e_N'(x))^2 dx + \int_0^1 (e_N(x))^2 dx}$,

where $e_N = u^N - u$.

Balanced norms

The error measured in the energy norm is too weak when used for singularly perturbed problems.

Balanced norms that correctly weight the error contribution need to be used.

One such norm is

- $||e_N||_{b(0,1)} = \sqrt{\varepsilon \int_0^1 (e_N'(x))^2 dx + \int_0^1 (e_N(x))^2 dx}.$

Generating meshes

- An *a priori* mesh:
  generated before the problem is solved.
- An *a posteriori* mesh:
  adapts an initial mesh using information gained by solving the equation.

An *a posteriori* mesh method

A monitor function is an arbitrary non-negative function defined on Ω, eg. [2]

$$M_{arc} = \sqrt{1 + (u'(x))^2}.$$

A mesh $\{x_i\}$ equidustributes $M(\cdot)$ when

$$\int_{x_{i-1}}^{x_i} M(x)dx = \frac{1}{N}\int_0^1 M(x)dx, \text{ for } i = 1, 2, ..., N.$$

Mesh generation using a balanced norm

1. Solve the equation on a uniform mesh.
2. Calculate the error measured in the balanced norm on each mesh interval.
3. Equidistribute

   $$M = (1 + \frac{1}{\alpha}||u^N||_b^2),$$

   where α is an intensity parameter. [1]
4. Repeat steps 2 and 3 until a preset stopping criteria is achieved.

Mesh generated using energy norm

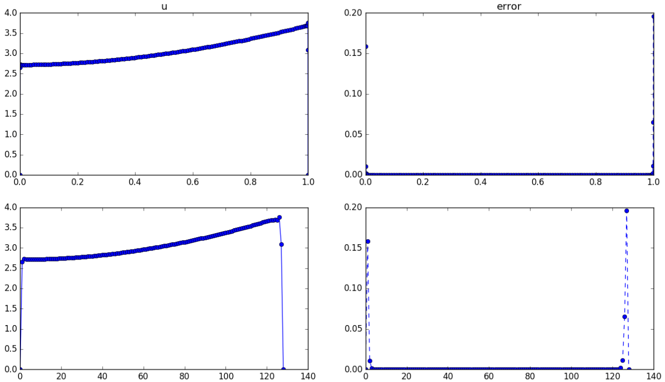

Figure: Outputs for $-(10^{-4})^2 u'' + (x+1)u = \exp(x+1)$, with $N = 128$

Mesh generated using balanced norm

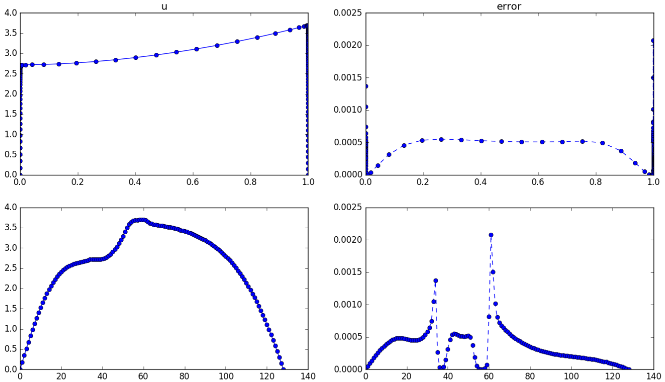

Figure: Outputs for $-(10^{-4})^2 u'' + (x+1)u = \exp(x+1)$, with $N = 128$

Future work

- Optimum intensity parameter.
- Efficiency.
- Convergence robustness.

References

[1] Weizhang Huang and Weiwei Sun.
Variational mesh adaptation ii: error estimates and monitor functions.
Journal of Computational Physics, 184(2):619–648, 2003.

[2] Natalia Kopteva and Martin Stynes.
A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem.
SIAM Journal on Numerical Analysis, 39(4):1446–1467, 2001.

Thank you for listening, any questions!