Discontinuous Galerkin Methods and FEniCS

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IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn

FEniCS PROJECT

PhD project

Balanced discontinuous Galerkin methods for fluid flow problems: design, implementation and application.

The project has three distinct phases

- Phase 1: Design a new set of FEMs for solving convection diffusion problems.
- Phase 2: Produce a working implementation of these in FEniCS.
- Phase 3: Apply these method to a physical river flow problem using data assimilation.

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Starting point - FYP

Become familiar with discontinuous Galerkin methods and their implimentation in FEniCS

- Read Chapter 1 of Béatrice Rivière's book "Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation" [2] in order to:
  - gain an understanding of discontinuous Galerkin (dG) methods
  - solve her one-dimensional Poisson's equation example in FEniCS and reproduce her results
- Extend these methods to a selection of ordinary and partial differential equations, including linear and non-linear and time-dependent convection-diffusion equations.

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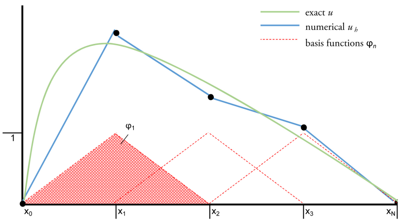
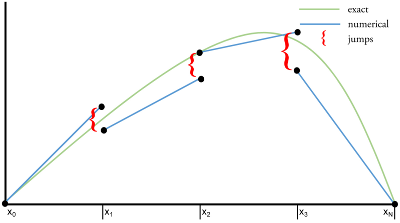
Finite Element Methods

- Finite element methods have been used since the 1950s.
- Galerkin methods are the most widely used (continuous Galerkin (cG)).
- discontinuous Galerkin (dG) methods are where my focus lies.

Comparison of cG and dG methods



cG solution



dG solution (before jumps penalised)

One-dimensional jumps and averages in v are respectively defined as

$$[v(x_n)] = v(x_n^-) - v(x_n^+),$$

and

$$\{v(x_n)\} = \frac{1}{2}\left(v(x_n^-) + v(x_n^+)\right).$$

Brief description of FEniCS

I used FEniCS [1] to implement the methods. FEniCS is a Python based collection of free software for solving partial differential equations. It provides access to low-level features, but automates other tasks.

Rivière's Poisson's example

$$
\begin{aligned}
-u''(x) = f(x) \text{ on } (0,1), \\
f(x) = \exp(-x^2)(4x^3 - 4x^2 - 6x + 2), \\
u(0) = 1, u(1) = 0, \\
u(x) = (1-x)\exp(-x^2).
\end{aligned}
\tag{1}
$$

The weak form of (1) is:

$$
-\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} u'' v dx = \int_0^1 f v dx.
$$

and integrating the left hand side by parts we get:

$$
-\sum_{n=0}^{N-1} u'v\Big|_{x_n}^{x_{n+1}} + \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} u'v' dx = \int_0^1 f v dx.
$$

Poisson's equation weak form

$$\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} u'v'dx - \sum_{n=0}^{N} \{u'(x_n)\}[v(x_n)] + \gamma \sum_{n=0}^{N} \{v'(x_n)\}[u(x_n)]$$

$$+ \sum_{n=0}^{N} \frac{\sigma^0}{h}[u(x_n)][v(x_n)] = \int_0^1 fvdx$$

$$+\gamma v'(x_N)u(x_N) - \gamma v'(x_0)u(x_0) - \frac{\sigma^0}{h_N}[u(x_N)][v(x_N)]$$

$$+ \frac{\sigma^0}{h_0}[u(x_0)][v(x_0)].$$

```
a = dot(grad(u), grad(v))*dx \
    - dot(avg(grad(u)), jump(v, n))*dS \
    + gamma*dot(jump(u, n), avg(grad(v)))*dS \
    + sigma/h_avg*dot(jump(u, n), jump(v, n))*dS \
    - dot(grad(u), v*n)*ds \
    + gamma*dot(u*n, grad(v))*ds \
    + (sigma/h)*u*v*ds
L = f*v*dx \
    + gamma*g*dot(grad(v), n)*ds \
    + (sigma/h)*v*g*ds
```

Poisson's equation weak form

$$\sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} u'v'dx - \sum_{n=0}^{N}\{u'(x_n)\}[v(x_n)] + \gamma\sum_{n=0}^{N}\{v'(x_n)\}[u(x_n)]$$

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    - dot(grad(u), v*n)*ds \
    + gamma*dot(u*n, grad(v))*ds \
    + (sigma/h)*u*v*ds
L = f*v*dx \
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Poisson's equation weak form

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L = f*v*dx \
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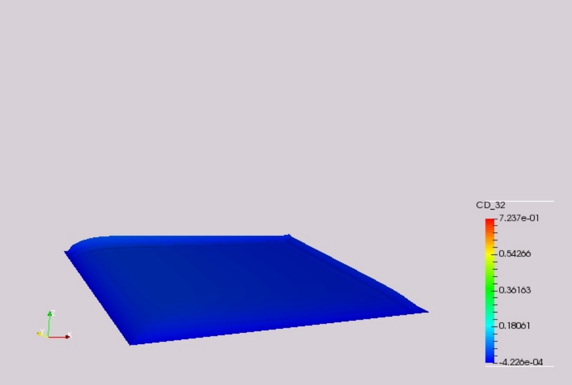
```

Other equations solved

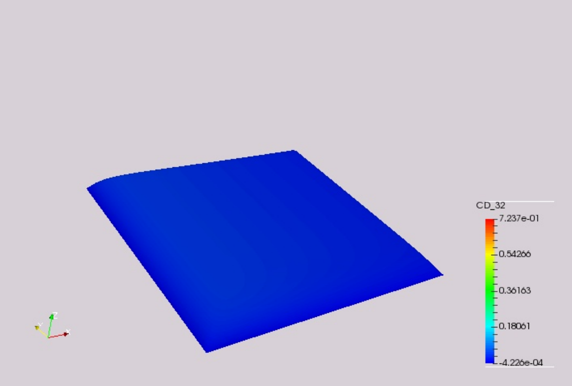
- I extended the methods to one-dimensional and two-dimensional convection-diffusion equations including the following components:
  - linear,
  - time-dependent and
  - non-linear.
- and then to a range of two-dimensional vector based convection-diffusion equations.

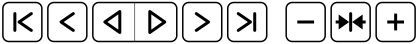
All problems were solved using a uniform mesh.

Solution to a convection-diffusion equation showing oscillations



Solution to $\frac{\delta u}{\delta t} - 0.01\Delta u + (1+x, 1+y)u = exp(-x^2)$, on $(0,0)(1,1)$, $u = u(x,y,t)$, $x(0) = x(1) = y(0) = y(1) = 0$, and N=32





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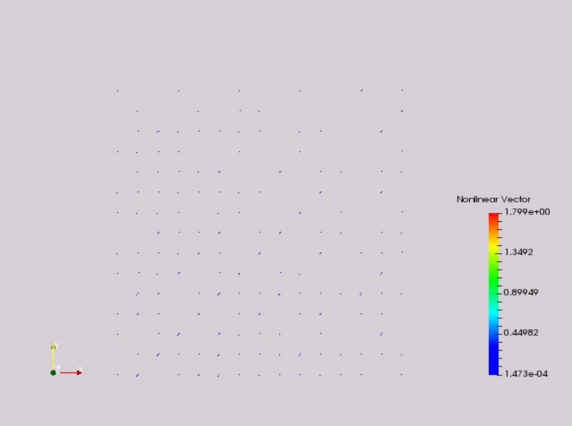
Solution to a vector based non-linear convection diffusion equation



Solution to $\frac{\delta \vec{u}}{\delta t} - 0.01\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} = (exp(-x), exp(-y))$, on $(0,0)(1,1)$, $\vec{u} = \vec{u}(x,y,t)$, $x(0) = x(1) = y(0) = y(1) = 0$ with N=16

Outcomes and Current work

Outcomes

- I now have an excellent understanding of the fundamentals of discontinuous Galerkin methods.
- I am competent in using the FEniCS software system to solve partial differential equations.

Current work

- Reviewing existing literature in the area.
- Investigating different methods of measuring errors.

Bibliography

Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al.
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Theory and implementation.