Evolving networks, an introduction

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Networks

"The important question is to explain how the interaction of a great number of people, each possessing only limited knowledge, will bring about an order that could only be achieved by deliberate direction taken by somebody who has the combined knowledge of all these individuals".

-Friedrich A. Hayek (1979)

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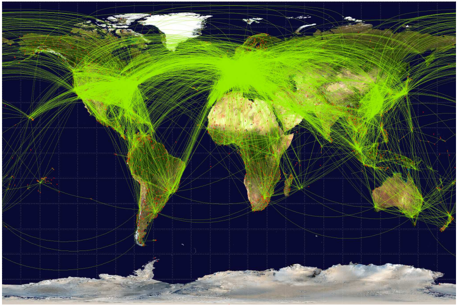

Mathematical networks

A *network* is a weighted graph, that is, a set of elements called *nodes* or *vertices*, which may be connected to one another via relational links (*edges*). To each node we assign a *state* s_i and to each edge a weight (or *gain*), $\sigma_{i,j}$.

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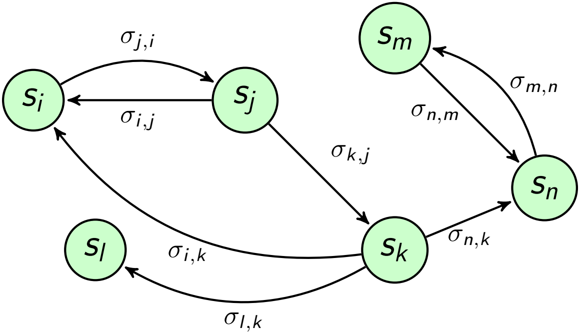

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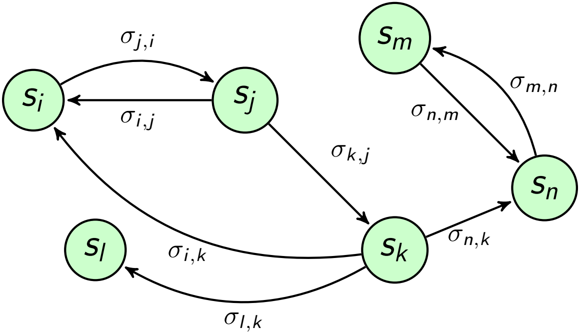

We want our states and gains to evolve until *consensus* is achieved.

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- friendships (gains) growing or deteriorating as people (nodes) cheer or vex one another.

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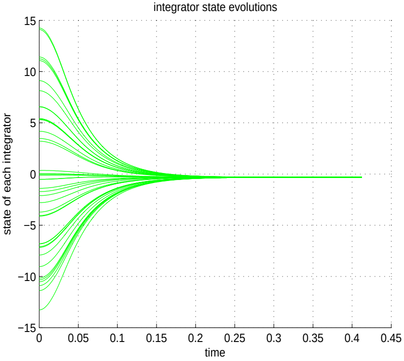

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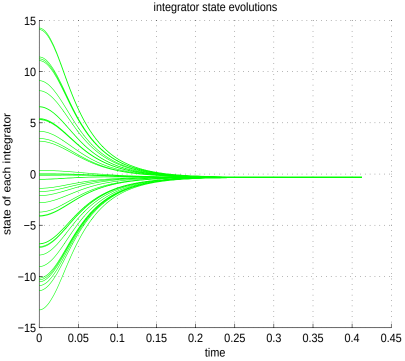

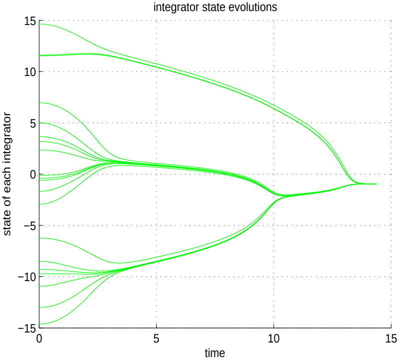

Differential equations

The state and gain evolutions are governed by a system of coupled differential equations. The general form being:

$$\underbrace{\frac{ds_i}{dt}}_{\text{state evolution}} = \underbrace{\xi(\sigma_{i,j}, s_i, s_j)}_{\text{influence of gains and other nodes}} .$$

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Let's now consider a particular evolution model.

Switch protocol

The *switch protocol* model allows the gains between nodes to grow until a certain proscribed threshold is reached, whereupon the gains (effectively) lock into that value. The states meanwhile, pull each other (via the gains) until a *consensus* is attained,

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$$\frac{d\sigma_{i,j}}{dt} = \begin{cases} \alpha h(s_i, s_j) e^{-\beta h(s_i, s_j)} & \text{if } \sigma_{i,j} < \tau, \\ 0 & \text{if } \sigma_{i,j} \geq \tau. \end{cases} \quad (1)$$

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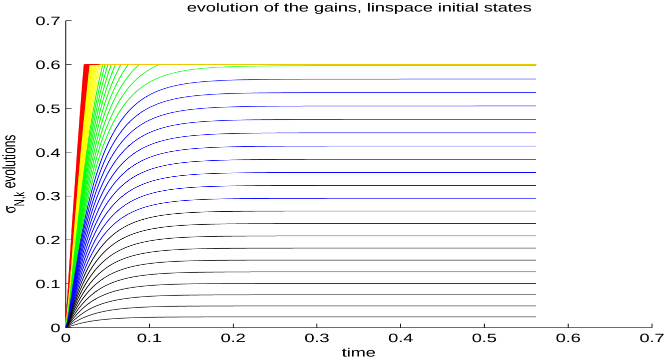
where α and β are rate parameters, h is some norm of the states and τ is the threshold where we want the gains to cease evolving.

The switch protocol gains grow and level off when the respective states come together. Notice the gains are capped by the threshold parameter τ. Here $\tau = 0.6$



Simulations

We generate networks using **Matlab** and investigate the evolutions of the states, gains and various features of the systems at consensus.

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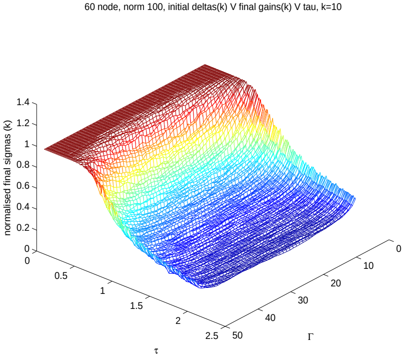

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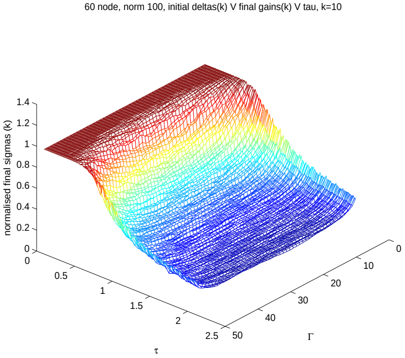
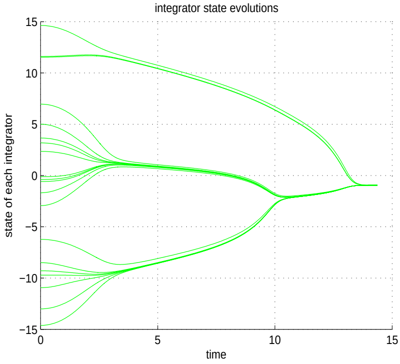
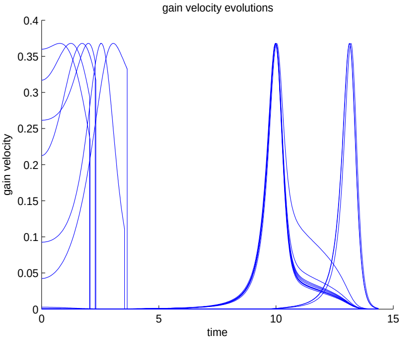




Gain evolutions



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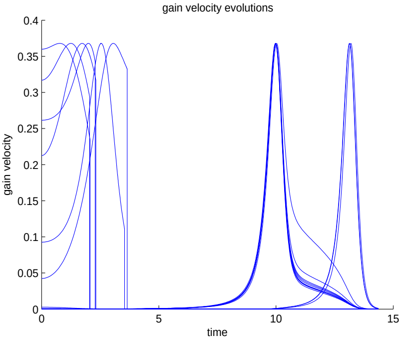

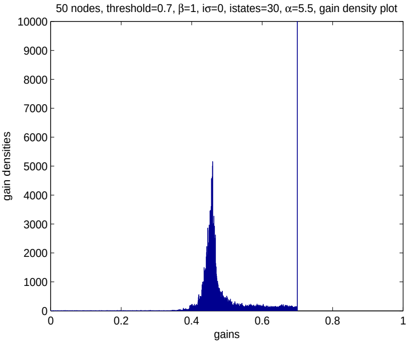

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Bimodal gain distributions are often observed for localised systems.

Reduced order approximation

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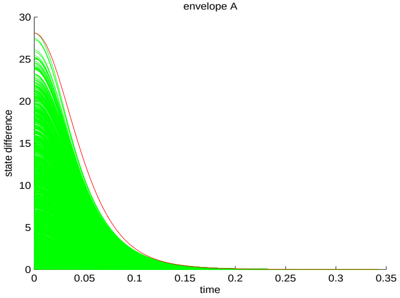

Envelope A (red), all other errors (green).

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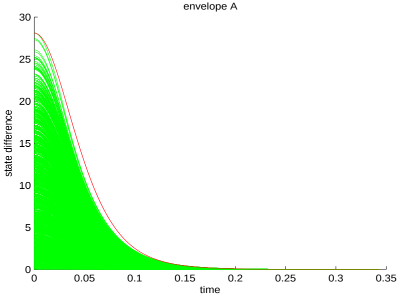

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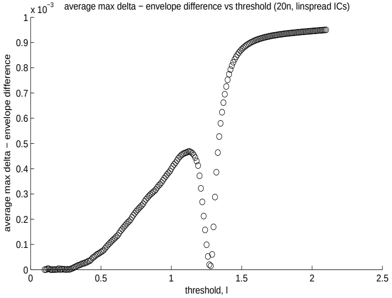

Absolute difference between (actual) max error and envelope.

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