Noise and Multistability in the Square Root Map

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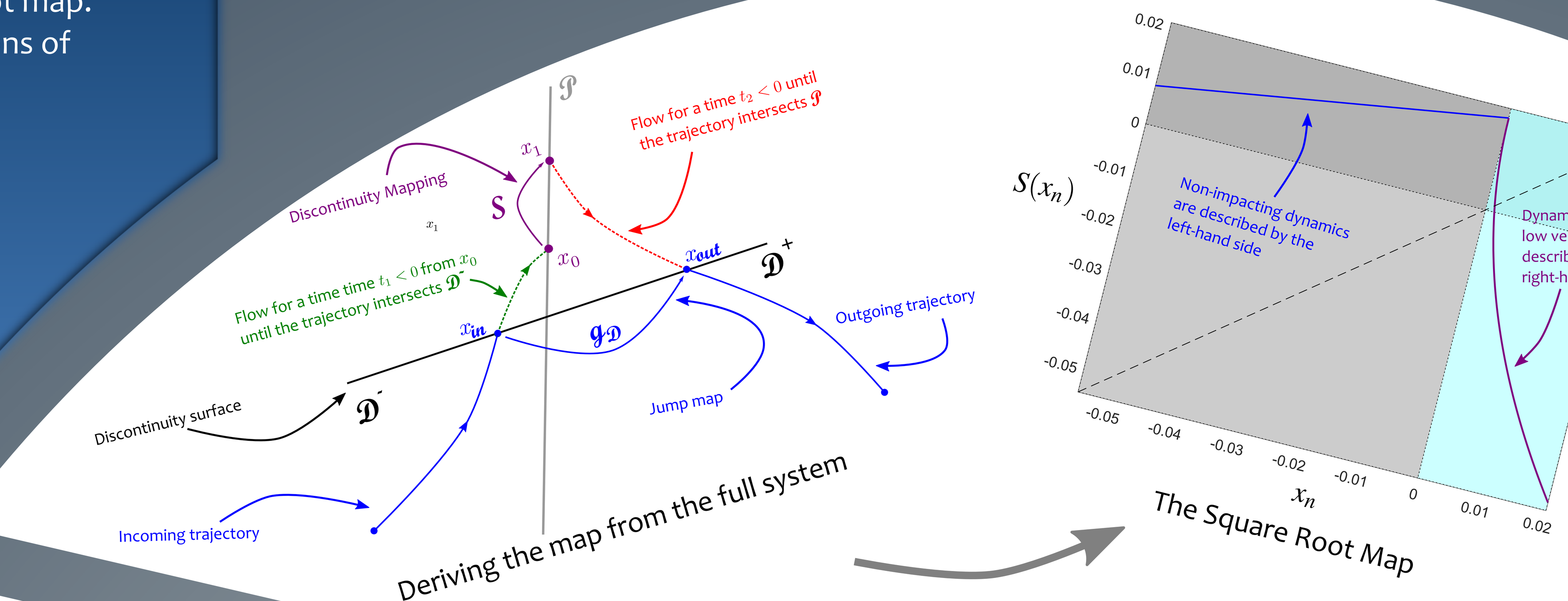
The Square Root Map

Many impacting systems, including impact oscillators undergoing low-velocity impacts, which are used to model systems arising in engineering such as moored ships impacting a dock or rattling gears, are described by a one-dimensional map known as the square root map. This continuous, nonsmooth map can be derived as an approximation for solutions of piecewise smooth differential equations near grazing impacts. We will write it as

$$x_{n+1} = S(x_n) = \begin{cases} \mu + bx_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0. \end{cases} \quad (1)$$

Nordmark [1],[2] uses an elegant argument to show that if $0 < b < \frac{1}{4}$ there are values of the bifurcation parameter $\mu > 0$ for which a single stable periodic orbit of period m exists for each $m = 2, 3, \ldots,$ and also such that two stable periodic orbits, one of period m, and the other of period $m+1$, exist for each m. These are the only possible attractors of the system except at bifurcation points. Here we will focus on period-2 and period-3 coexistence.

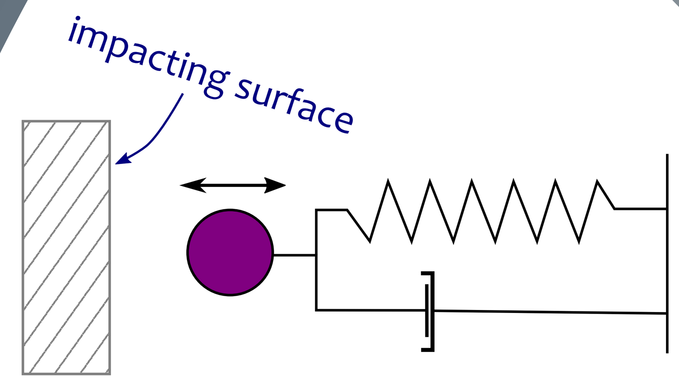


Consider the return map on $\mathcal{P}$ that transversally intersects the discontinuity surface at the point corresponding to zero-velocity impacts. In the absence of impacts the map is trivial. For points on $\mathcal{P}$ in the region beyond the discontinuity surface we do the following:

1. flow for a time $t_1 < 0$ with the vector field until reaching $\mathcal{D}$
2. apply the jump map
3. flow for a time $t_2 < 0$ with the vector field until reaching $\mathcal{P}$

Adding Noise

Our interest is in the qualitative behaviour of the square root map in the presence of additive white noise. In particular we focus on the effect of noise of varying amplitudes on systems with values of μ in, or close to, the intervals of multistability, for which stable periodic orbits of period m and $m+1$ coexist. In these regions complicated deterministic structures interact with noise to produce interesting dynamics.



A forced impact oscillator

Varying the Noise Amplitude Δ

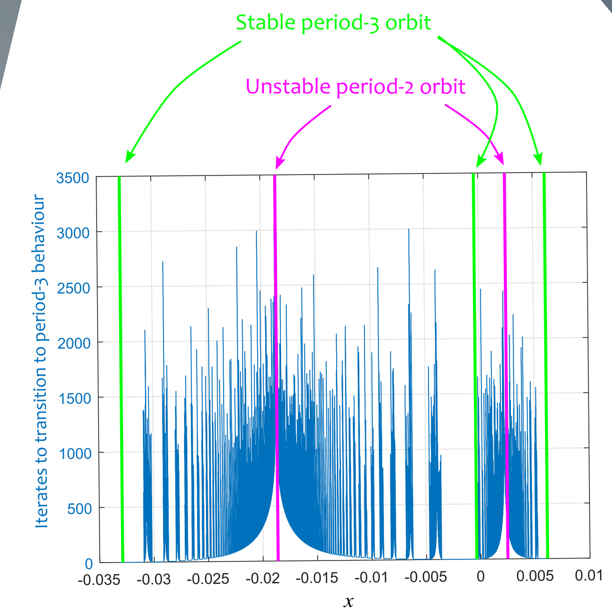
We consider the square root map with additive gaussian white noise of amplitude Δ:

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0, \end{cases} \quad (2)$$

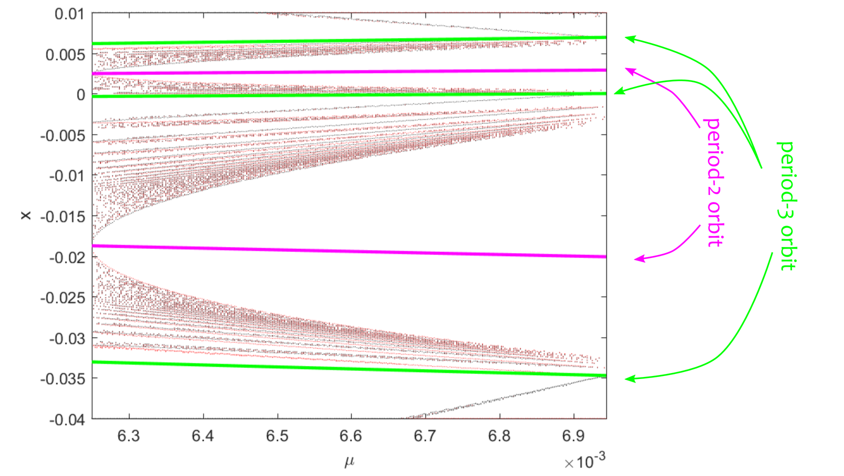
where $\xi_n \sim N(0, \Delta^2)$. We are interested in the qualitative behaviour of the map over time. Instead of looking at the value of individual iterates we will denote iterates on the right as Rs and iterates on the left as Ls, corresponding to low-velocity impacts and non-impacting dynamics, respectively. For $0 < b < \frac{1}{4}$ in the deterministic system periodic orbits of period m take the form $(RL^{m-1})^{\infty}$.

Focusing on values of μ in, or close to, the interval of coexistence for RL and RLL attractors we find that adding noise of low amplitude to the system causes the interval of coexistence to effectively shrink. Near the threshold for RL stability low amplitude noise can push all dynamics into the basin of attraction of the RLL attractor. However, increasing the noise amplitude we find this trend reverses, in fact we even begin to see persistent RL behaviour in the region where the RL orbit is unstable.

The nonsmooth nature of the square root map creates complicated deterministic structures.

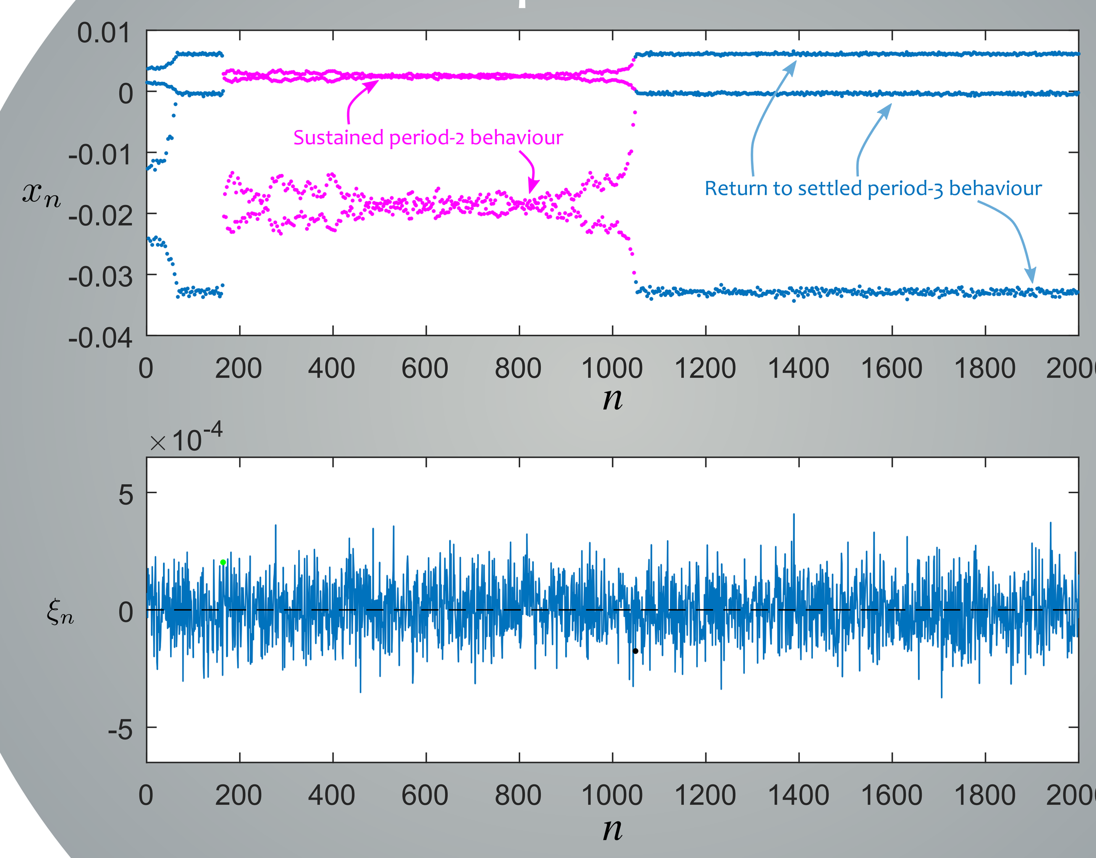


For values of μ close to the interval of multistability, where the period-2 orbit is unstable, we see that the relationship between the time taken to transition to period-3 behaviour and our initial condition is very complicated.



On the interval of multistability where both the period-2 and the period-3 attractors are stable the basins of attraction have a fine *riddled* structure.

A Sample Transition



Associated Noisy Signal

The Transition Mechanism

Perhaps the most interesting phenomenon that we have observed is the potential for repeated intervals of persistent RL dynamics in a noisy system with μ such that the period-2 orbit is unstable in the corresponding deterministic system.

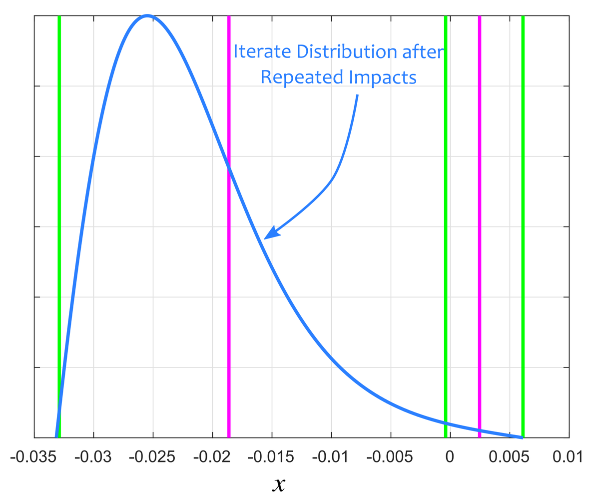
We have observed that the noise-induced transition between RLL and RL behaviour in this case takes the following symbolic form:

$$RLL \ldots RLLRLRRLRL \ldots RL.$$

The most significant feature of the transition is the repeated R (RR), corresponding to repeated low-velocity impacts. This is triggered by the error due to noise pushing the second left iterate of the period-3 orbit onto the right.

These repeated low-velocity impacts allow the dynamics to be pushed into the region of phase space with slow dynamics, in the vicinity of the unstable period-2 orbit of the deterministic system.

The system can then take a significant number of iterates to transition back to RLL behaviour. In fact, once close to the unstable orbit noise can have a stabilising effect, pushing the dynamics back towards the unstable period-2 orbit on the transition back to RLL behaviour.

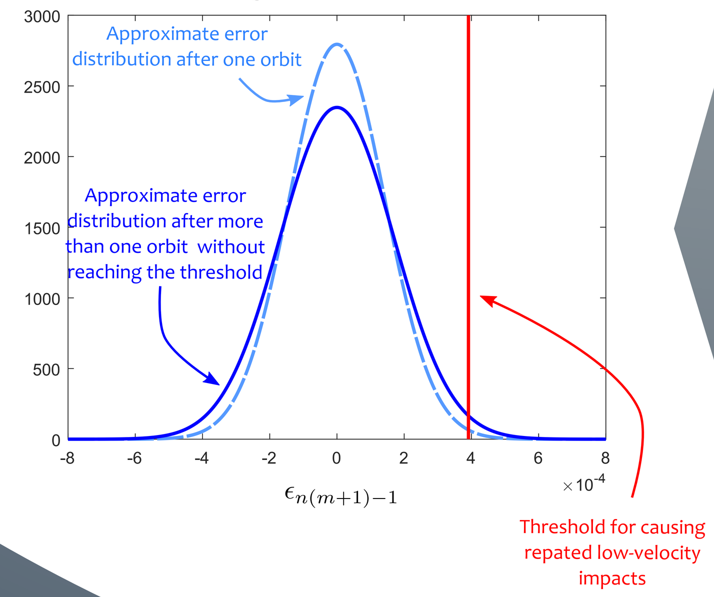


Repeated low velocity impacts concentrate trajectories with errors above the threshold around the unstable period-2 orbit . . .

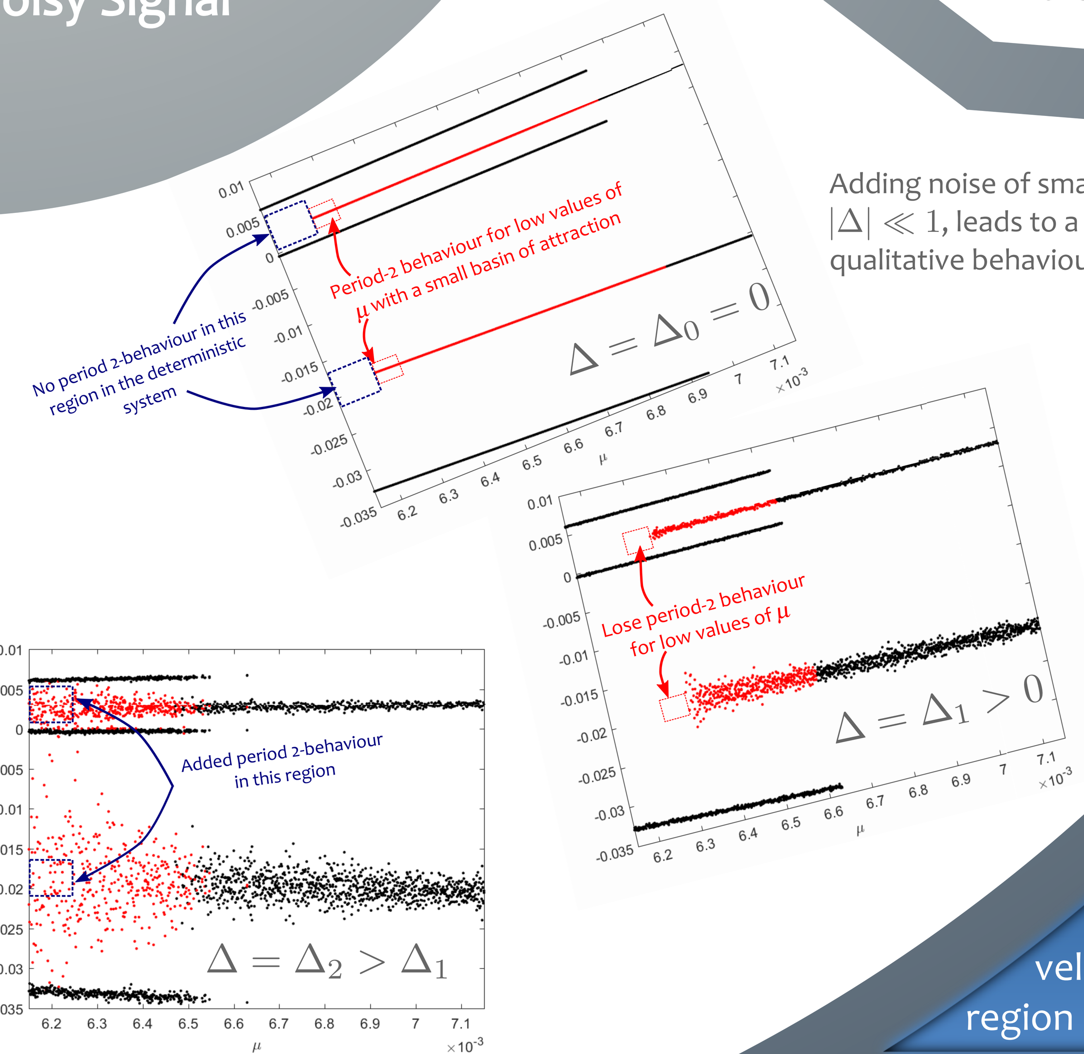
. . . causing transitions like the one seen in the centre of the poster.

Consider a value of μ close to the interval of multistability for period-2 and period-3 but where the period-2 orbit is unstable.

To cause repeated low-velocity impacts after starting in the deterministc period-3 orbit, the error must be above the marked threshold pushing the last left iterate of the period-3 orbit onto the right.



Adding noise of small but increasing amplitude, $|\Delta| \ll 1$, leads to a non-monotonic response in qualitative behaviour.



Why Consider Noise and Nonsmoothness?

Historically mathematicians have made widespread use of smooth, deterministic mathematical models to describe real-world phenomena. These models present a simplified view of the world where, on one hand, the evolution of systems is always smooth and exhibits no interruptions such as impacts, switches, slides or jumps and, on the other hand, the future of any system is completely determined by its present state.

However, when modelling many real-world systems one or both of these simplifications may not hold. For example, mechanical systems involving impacts or friction and electrical systems with switches behave in a nonsmooth manner and more complex systems such as the world's climate have also been modelled using nonsmooth models. Furthermore, it has been shown that a level of randomness or noise is ubiquitous in real-world systems.

Independently, both noise and nonsmoothness have been shown to be the drivers of significant changes in qualitative behaviour. However, the combined effect of noise and nonsmoothness has seen limited research.

Generalising

Although we have focused on the case of period-2 and 3 coexistence here, similar results hold for the period-m and $m+1$ coexistence. In particular, a non-monotonic relationship between noise amplitude and qualitative behaviour exists. Transitions from RL^m to RL^{m-1} behaviour in the region where the RL^{m-1} orbit is unstable take the following form, with $2 \leq k \leq m$:

$$RL^m \ldots RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1} \ldots RL^{m-1}.$$

The most significant feature of this transition is again the repetition of low-velocity impacts in quick succession ($RL^{k-2}R$), forcing the dynamics into the region of phase space close to the unstable period-m orbit.

References

[1] A.B. Nordmark (1991) Non-periodic motion caused by grazing incidence in an impact oscillator, *J. Sound Vib.* **145** 279–297.
[2] A.B. Nordmark (1997) Universal limit mapping in grazing bifurcations, *Phys. Rev. E* **55** 266–270.
[3] F. Breuer (2010) Poster Template, http://blog.felixbreuer.net/2010/10/24/poster.html, (CC BY-SA 3.0 License).