A NOISY SALTATION MATRIX

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1 Introduction

In a smooth dynamical system given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the characteristics of a given reference trajectory with starting point $\mathbf{x}_0^{\text{ref}}$ can be determined, to lowest order, by examining the linearised system about the reference trajectory. In other words, we can approximate the deviations of trajectories with starting points $\mathbf{x}_0 \approx \mathbf{x}_0^{\text{ref}}$ after a given time t by multiplying $\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)$, the *fundamental solution matrix* of the reference trajectory, by the initial deviations of the trajectories.

$$\phi(\mathbf{x}_0, t) - \phi(\mathbf{x}_0^{\text{ref}}, t) = \phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, t)(\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}) + \mathcal{O}(\|\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}\|). \tag{1}$$

This analysis method cannot directly be used in nonsmooth systems as it stands since the vector field $\mathbf{f}$ is not everywhere differentiable, or the flow function $\phi(\mathbf{x}_0^{\text{ref}}, t)$ is not continuous. To account for this we derive the *zero-time discontinuity mapping* $\mathbf{D}$ associated with the discontinuity boundary, i.e. we find the map $\mathbf{D}$ such that when time $t = T$ is greater than the length of time it takes the trajectory starting at $\mathbf{x}_0$ to cross the boundary

$$\phi(\mathbf{x}_0, T) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), T - t_{\text{ref}}), \tag{2}$$

where ϕ_i are the flows corresponding to the vector fields $\mathbf{f}_i$ on either side of the discontinuity boundary $\mathcal{D}$ and t_{ref} is the time of flight from $\mathbf{x}_0^{\text{ref}}$ to the boundary. The derivative $\mathbf{D}_{\mathbf{x}}$ of the mapping $\mathbf{D}$ is known as the *saltation matrix* and its properties can tell us how the crossing of the discontinuity boundary affects the deviations of trajectories from a reference trajectory.

We are interested in deriving the saltation matrix of the system in the case where the vector fields on either side of the discontinuity boundary are deterministic but the boundary varies randomly in time.

2 Deriving $\mathbf{D}$ and $\mathbf{D}_{\mathbf{x}}$ for a Deterministic System

In the deterministic case we consider the system shown in Figure 1 where the jump map $\mathbf{j}$ is applied on the discontinuity boundary $\mathcal{D}$ given by $h(\mathbf{x}) = 0$. Let $t(\mathbf{x})$ be the (possibly negative) time of flight from $\mathbf{x}$ to the boundary.
To calculate $\mathbf{D}$ for a point $\mathbf{x} = \phi_1(\mathbf{x}_0, t_{\text{ref}})$, where $\mathbf{x} \approx \mathbf{x}_0$, we take the following steps:

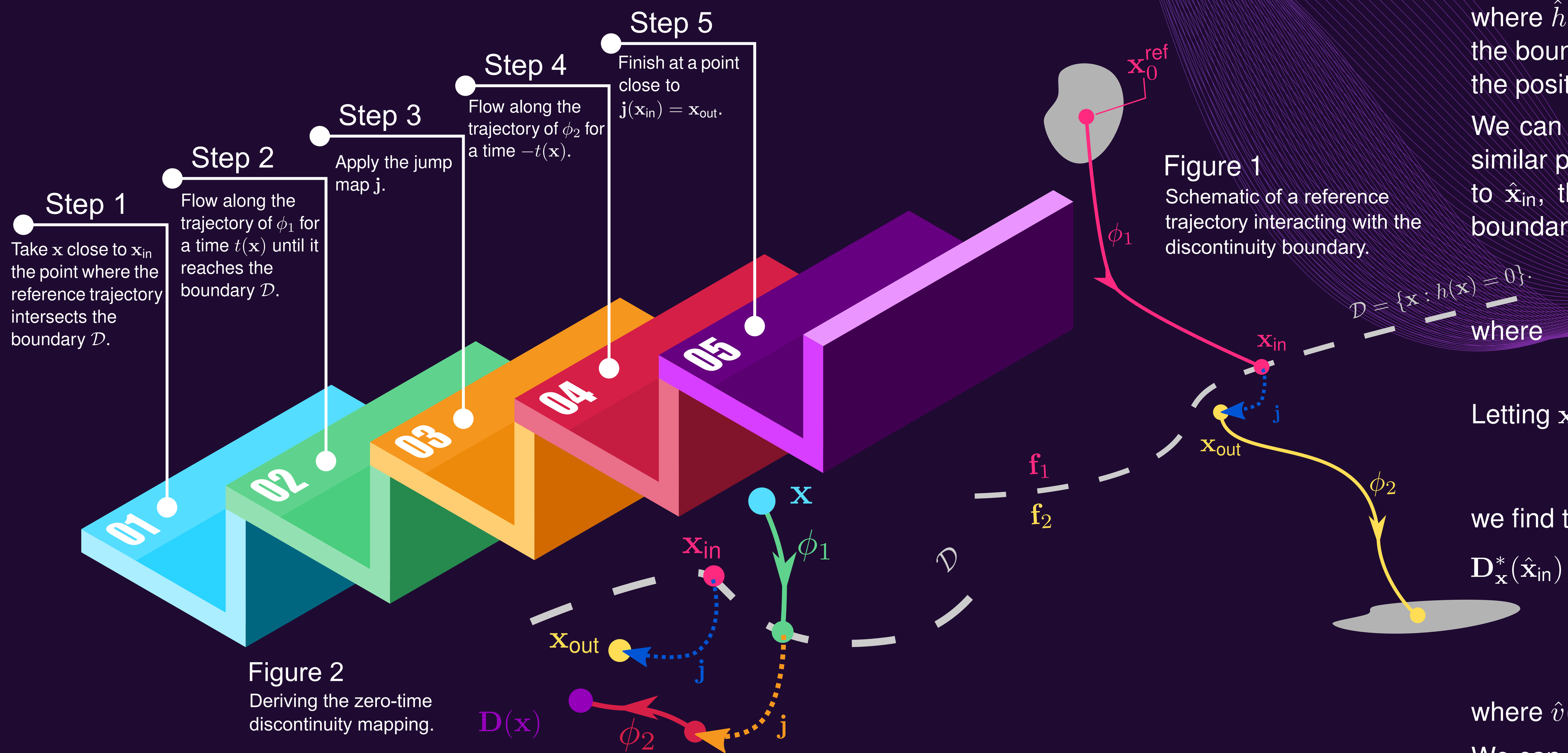

Figure 1 Schematic of a reference trajectory interacting with the discontinuity boundary.

Figure 2 Deriving the zero-time discontinuity mapping.

and so

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x})). \tag{3}$$

The Jacobian of $\mathbf{D}$ evaluated at $\mathbf{x}_{\text{in}}$ is given by

$$\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) = \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) + \frac{(\mathbf{f}_{\text{out}} - \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}})h_{\mathbf{x}}(\mathbf{x}_{\text{in}})}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}}, \tag{4}$$

where $\mathbf{f}_{\text{in}} = \mathbf{f}_1(\mathbf{x}_{\text{in}})$ and $\mathbf{f}_{\text{out}} = \mathbf{f}_2(\mathbf{x}_{\text{out}})$. In the case where h and $\mathbf{j}$ are explicitly time-dependent this becomes

$$\mathbf{D}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) = \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}) + \frac{(\mathbf{f}_{\text{out}} - (\mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}} + \mathbf{j}_t(\mathbf{x}_{\text{in}}, t_{\text{ref}})))h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})}{h_t(\mathbf{x}_{\text{in}}, t_{\text{ref}}) + h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}}}. \tag{5}$$

3 Why Consider Noise?

Traditionally mathematicians have used smooth deterministic models to model the real world. These models present a simplified view of the world where the evolution of systems exhibits no interruptions such as impacts, switches, or jumps and there is no uncertainty (or noise) present. However, independently, both nonsmoothness and noise have been shown to drive significant changes in the behaviour of a model. It is therefore important to investigate and understand how the inclusion of both nonsmoothness and noise can affect the behaviour of a model.

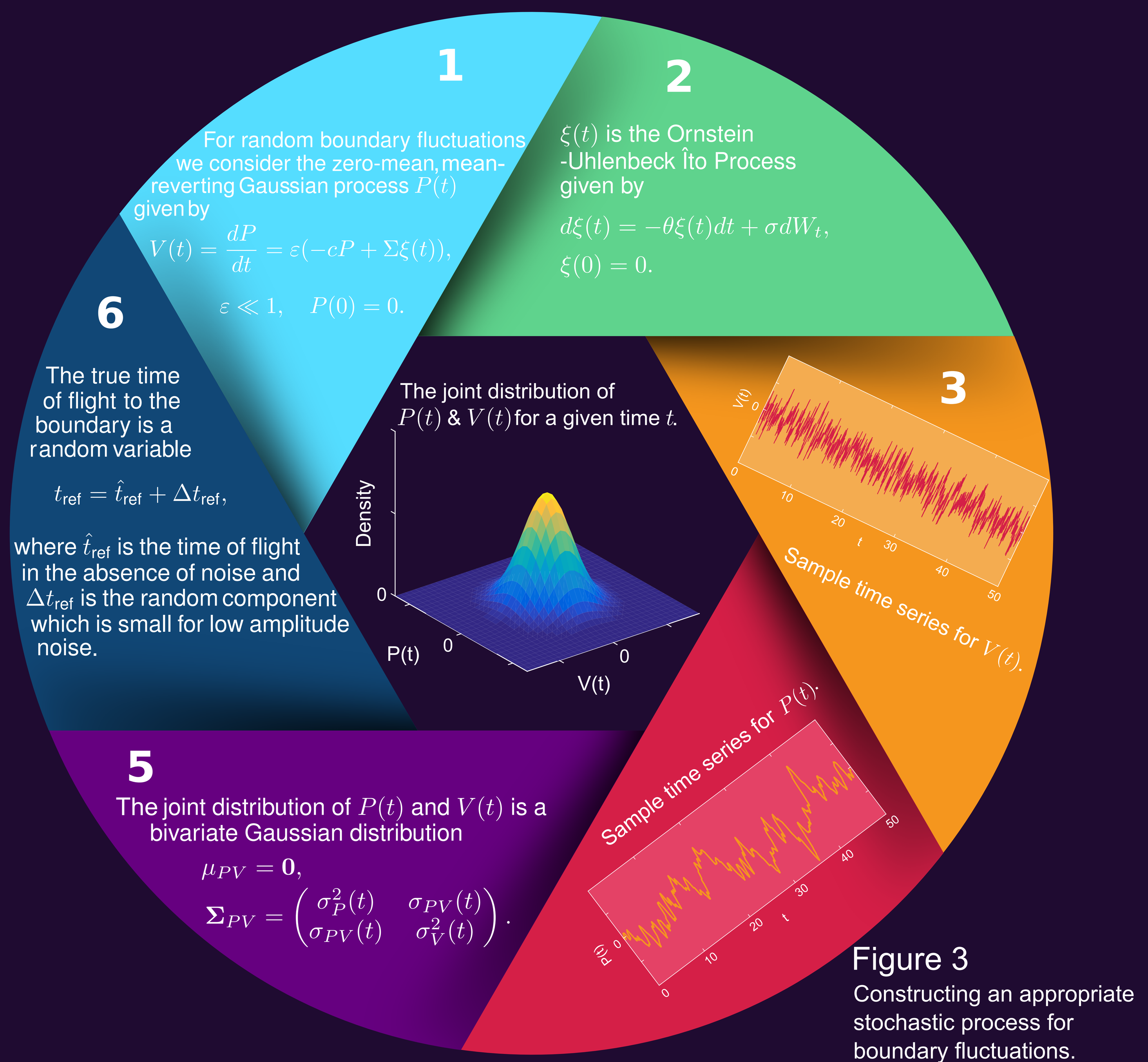

Figure 3 Constructing an appropriate stochastic process for boundary fluctuations.

4 The Noisy Case

Let us consider the system with a boundary whose motion has a random component. The discontinuity boundary in this case is given by

$$h(\mathbf{x}, t) = \hat{h}(\mathbf{x}) - \hat{P}(t) - P(t) = 0, \tag{6}$$

where $\hat{h}$ is a deterministic function, $\hat{P}(t)$ represents the deterministic movement of the boundary and $P(t)$ is a stochastic process representing random fluctuations in the position of the boundary, for example the process derived in Figure 3.

We can now construct a new discontinuity mapping in the noisy case following a similar process to the deterministic case shown in Figure 2. Taking a point $\mathbf{x}$ close to $\hat{\mathbf{x}}_{\text{in}}$, the point of intersection of the reference trajectory with the discontinuity boundary in the absence of fluctuations, we set

$$\mathbf{D}^*(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t^*(\mathbf{x})), -t^*(\mathbf{x})). \tag{7}$$

where

$$t^*(\mathbf{x}) = t(\phi_1(\mathbf{x}, \Delta t_{\text{ref}})) + \Delta t_{\text{ref}}. \tag{8}$$

Letting $\mathbf{x}_{\text{in}}$, $\mathbf{f}_{\text{in}}$ and $\mathbf{f}_{\text{out}}$ be the random variables given by

$$\mathbf{x}_{\text{in}} = \phi_1(\mathbf{x}_0^{\text{ref}}, t_{\text{ref}}), \quad \mathbf{f}_{\text{in}} = \mathbf{f}_1(\mathbf{x}_{\text{in}}), \quad \mathbf{f}_{\text{out}} = \mathbf{f}_2(\mathbf{j}(\mathbf{x}_{\text{in}})), \tag{9}$$

we find that the appropriate saltation matrix is the random matrix given by

$$\mathbf{D}_{\mathbf{x}}^*(\hat{\mathbf{x}}_{\text{in}}) = \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, -\Delta t_{\text{ref}})\mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\phi_{1,\mathbf{x}}(\mathbf{x}_{\text{in}}, \Delta t_{\text{ref}}) + \frac{[\mathbf{f}_2(\mathbf{D}^*(\hat{\mathbf{x}}_{\text{in}})) - \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, -\Delta t_{\text{ref}})(\mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}} + \mathbf{j}_t(\mathbf{x}_{\text{in}}, t_{\text{ref}}))]\, h_{\mathbf{x}}(\mathbf{x}_{\text{in}})}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}} - \hat{v}(t_{\text{ref}}) - V(t_{\text{ref}}|P = P^*)}, \tag{10}$$

where $\hat{v}(t_{\text{ref}}) = \frac{d\hat{P}}{dt}$ and $P^* = \hat{h}(\mathbf{x}_{\text{in}}) - \hat{P}(t_{\text{ref}})$.

We can now write the fundamental solution matrix of a trajectory with initial condition $\mathbf{x}_0^{\text{ref}}$ after time $T > \hat{t}_{\text{ref}}$, where $\hat{t}_{\text{ref}}$ is the time it takes for the trajectory to cross the boundary in the absence of noise, as

$$\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, T) = \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, T - \hat{t}_{\text{ref}})\mathbf{D}_{\mathbf{x}}^*(\hat{\mathbf{x}}_{\text{in}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}}). \tag{11}$$

We see that the entire effect of both the discontinuity and the randomness is contained within $\mathbf{D}_{\mathbf{x}}^*(\hat{\mathbf{x}}_{\text{in}})$ since both $\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}})$ and $\phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, T - \hat{t}_{\text{ref}})$ are entirely smooth and deterministic.

In systems which are nonlinear it can be difficult to compute the exact distribution of $\mathbf{x}_{\text{in}}$ when we are not evaluating exactly at $\hat{\mathbf{x}}_{\text{in}}$, the deterministic point of intersection with the boundary. In these cases we can take a first order approximation of $\mathbf{x}_{\text{in}}$

$$\tilde{\mathbf{x}}_{\text{in}} = \hat{\mathbf{x}}_{\text{in}} + \hat{\mathbf{f}}_{\text{in}}\Delta t_{\text{ref}}. \tag{12}$$

References

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2. H. Dankowicz and P.T. Piiroinen, *Exploiting discontinuities for stabilization of recurrent motions*, Dynamical Systems, vol. 17, no. 4, pp. 317-342, 2002.
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5 Motivating Example - A Bouncing Ball System

Consider the deterministic system where a ball falling with constant acceleration due to gravity g, bounces inelastically on a floor oscillating sinusoidally with amplitude γ and frequency ω. At impact the relative velocity between the floor and the ball is instantaneously reversed and is scaled by a factor $r \in (0, 1)$, known as the coefficient of restitution. In our notation such a system is given by

$$\mathbf{x} = (x_1, x_2)^T, \qquad \mathbf{f}_1 = \mathbf{f}_2 = (x_2, g)^T, \tag{13}$$
$$h(\mathbf{x}, t) = x_1 - \gamma\sin(\omega t), \qquad \mathbf{j}(\mathbf{x}) = (x_1, -rx_2 + (r+1)\gamma\omega\cos(\omega t))^T.$$

This system has a large family of periodic orbits of period $T = \frac{2n\pi}{\omega}$, where $n \in \mathbb{N}$, with one impact per period which exist provided $\left|\frac{\pi n g}{\gamma\omega^2}\left(\frac{r-1}{r+1}\right)\right| < 1$. Using the deterministic saltation matrix (cf. (5)) to calculate the fundamental solution matrix of these periodic orbits, we find that the eigenvalues of the fundamental solution matrix are less than 1 in magnitude and therefore the periodic orbit is stable provided

$$-\gamma\omega^2\sin(\omega\tau') > \frac{2g(r^2+1)}{(r+1)^2}, \tag{14}$$

where τ' is the time of impact of the periodic orbit $\bmod \frac{2\pi}{\omega}$.

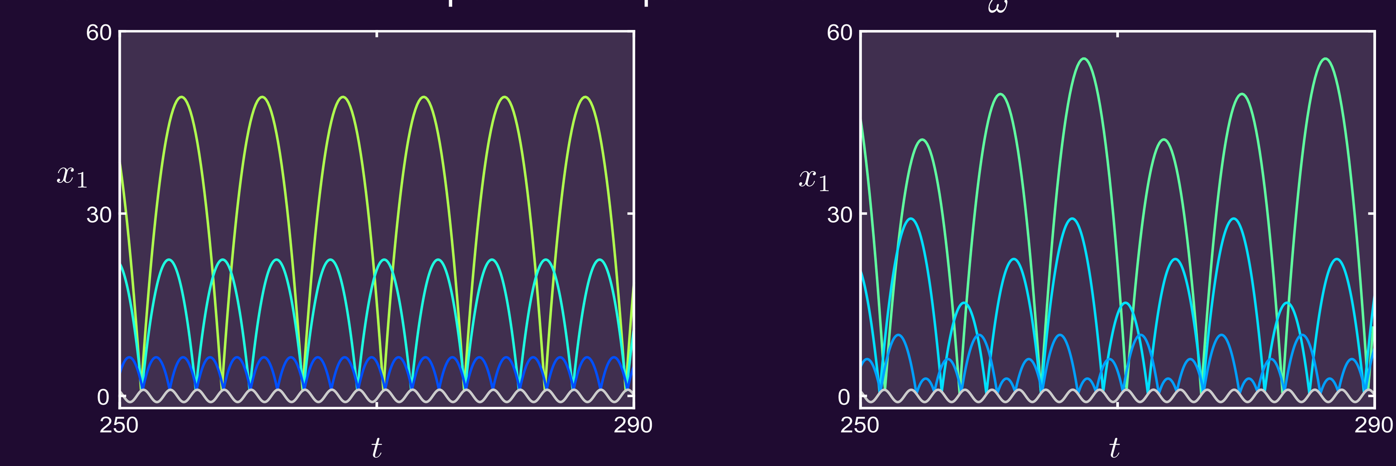
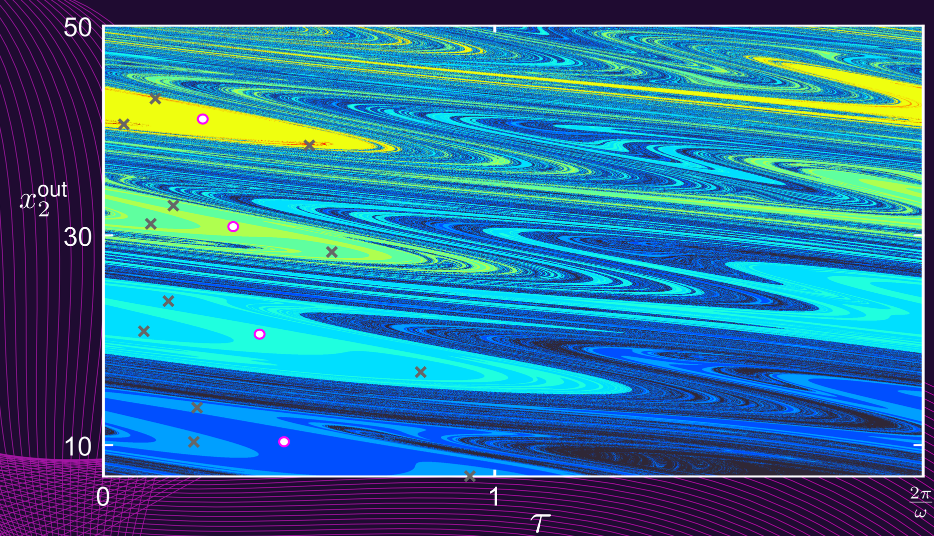


Figure 4 Six coexisting stable periodic orbits in the bouncing ball system when $\gamma = 1$ and $\omega = 3$.

This system can have several other coexisting attractors depending on our choice of γ and ω. For example in Figure 4 we plot six of the systems coexisting attractors when $\gamma = 1$ and $\omega = 3$, the periodic orbits that impact once per period for $n = 1, 2, 3$, and three associated periodic orbits of period $\frac{6n\pi}{\omega}$ with three impacts per period.

In Figure 5 we plot the basins of attraction of all attractors in (x_2^{out}, τ)-space when $\gamma = 1$ and $\omega = 3$. Here x_2^{out} is the post-impact velocity and τ is the time of impact $\bmod \frac{2\pi}{\omega}$. Due to the complicated intermingled structure of these basins the behaviour of the system has the potential to be highly sensitive to the addition of a noisy component to the floor's oscillation.



Periodic orbit - one impact per period, $T = \frac{2n\pi}{\omega}$.
Periodic orbit - three impacts per period, $T = \frac{6n\pi}{\omega}$.

Each colour in the state-space corresponds to a different basin of attraction. Any initial condition in a given basin of attraction will eventually be absorbed by the attractor in that basin.
For example, the dynamics plotted in Figure 4 are colour coded by their corresponding basins of attraction.
Black corresponds to initial conditions absorbed by attractors outside the region shown.

Figure 5 Basins of attraction for the bouncing ball system with $\gamma = 1$ and $\omega = 3$.

6 Future Work

Suppose we now introduce a noisy component to the floor's oscillation by letting

$$h(\mathbf{x}, t) = x_1 - \gamma\sin(\omega t) - P(t), \tag{15}$$

where $P(t)$ is a suitable stochastic process. Referring to (9) and (10) we can now calculate the fundamental solution matrices

$$\phi_{\mathbf{x}}\left(\mathbf{x}_0^{\text{ref}}, \frac{2n\pi}{\omega}\right) = \begin{pmatrix} -r - \Delta t_{\text{ref}}\Gamma & (1-r)\Delta t_{\text{ref}} - \frac{2n\pi}{\omega}(r + \Delta t_{\text{ref}}\Gamma) \\ \Gamma & \frac{2\pi n}{\omega}\Gamma - r \end{pmatrix}, \tag{16}$$

of the periodic orbits with one impact per period. Here

$$\Gamma = (r+1)\frac{g - (-\gamma\omega^2\sin(\omega t_{\text{ref}}) + A(t_{\text{ref}}|P = P^*))}{\hat{x}_2^{\text{in}} + g\Delta t_{\text{ref}} - (\hat{v}(t_{\text{ref}}) + V(t_{\text{ref}}|P = P^*))}, \tag{17}$$

where $\hat{x}_2^{\text{in}}$ is the incoming impact velocity of the deterministic orbit and $A(t) = \frac{dV}{dt}$ is the stochastic component of the floors acceleration.

Our aim is to analyse the characteristics of the random fundamental solution matrices of the attractors of this system, and other nonsmooth systems, in the presence of noise. These characteristics have the potential to give us great insight into how the introduction of noise affects the dynamics of this system. In particular the distributions of the eigenvalues and eigenvectors should allow us to understand how the introduction of noise affects the stability of attractors and induces transitions between different types of dynamics