Noise & Multistability in the Square Root Map

Eoghan Staunton, Petri T. Piiroinen

7 December, 2016

NUI Galway
OÉ Gaillimh

IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn

Noise and Nonsmoothness in Dynamical Systems

Both noise and nonsmoothness have been shown to independently be the drivers of significant changes in qualitative behaviour.

- Nonsmooth systems - qualitative changes in the behavior of the system under parameter variation that do not occur in the smooth setting.
- Adding noise to (smooth) systems - does more than just blur the outcome of the system in the absence of noise

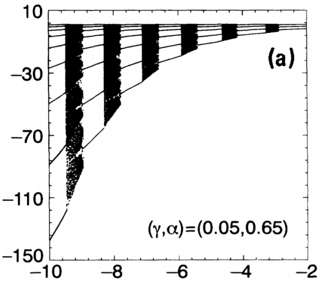


Figure: From Chin et. al, [CONG94].

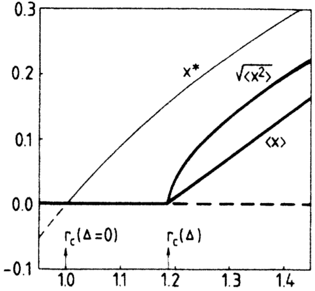
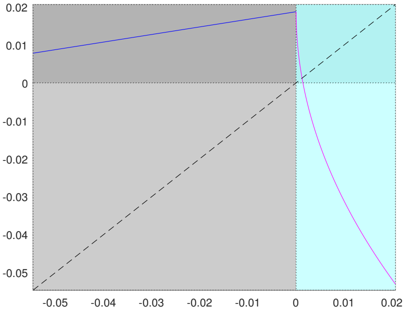


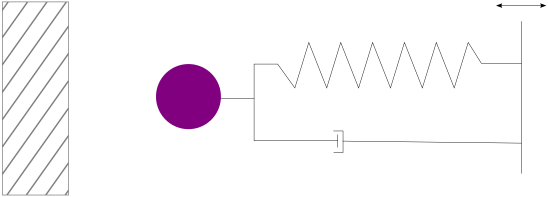
Figure: From Linz and Lücke, [LL86].

The Square Root Map

Many impacting systems, including impact oscillators, are described by a 1-D map known as the square root map.

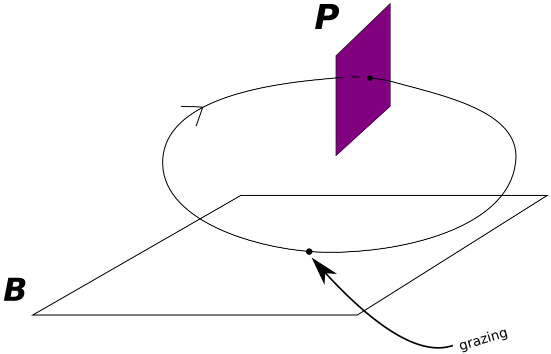
$$x_{n+1} = S(x_n) = \begin{cases} \mu + b x_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases}$$





The Square Root Map

This continuous, nonsmooth map can be derived as an approximation for solutions of piecewise smooth differential equations near certain types of grazing bifurcation.

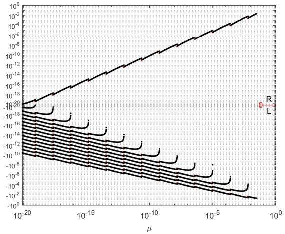


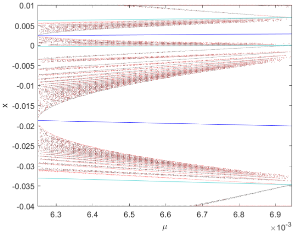
Multistability In the Square Root Map

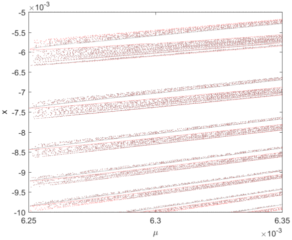
If $0 < b < \frac{1}{4}$ there are values of $\mu > 0$ for which

- a single stable periodic orbit of period m, with code $(RL^{m-1})^{\infty}$, exists for each $m = 2, 3, \ldots$
- two stable periodic orbits, one of period m, with code $(RL^{m-1})^{\infty}$, and the other of period $m+1$, with code $(RL^{m})^{\infty}$, exist for each $m = 2, 3, \ldots$

These are the only possible attractors of the system except at bifurcation points.







Types of Noise

In two separate papers Simpson, Hogan and Kuske and Simpson and Kuske make a careful analysis of how noise in impacting systems manifests in the map. They conclude that there are several different models. We focus on two of the simpler models with Gaussian white noise.

1. **Additive Noise**

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0 \end{cases} \tag{1}$$

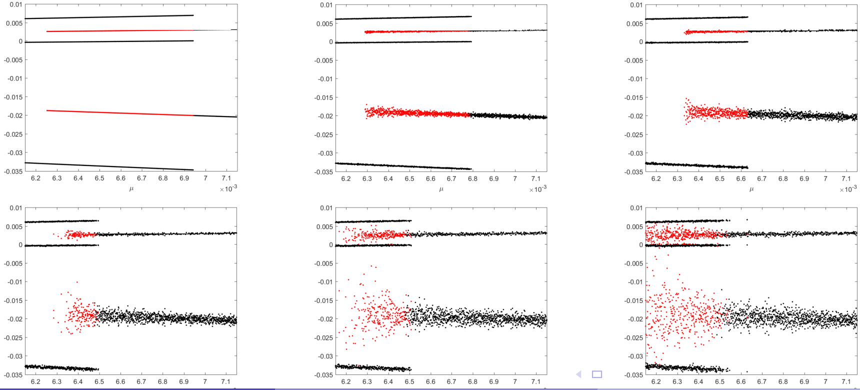
2. **Parametric Noise**

$$x_{n+1} = S_p(x_n) = \begin{cases} \mu + bx_n & \text{if } x_n < 0 \\ \mu - \left(a + \frac{1}{2}\xi_n\right)\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases} \tag{2}$$

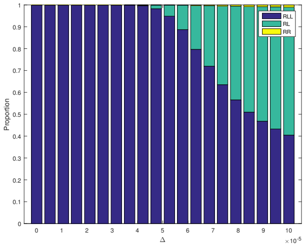
The Effect of Noise

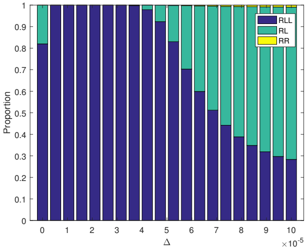
My work thus far has focused on phase space sensitivity for period m and $m+1$ coexistence, investigating a shift of the proportion of points going to one behaviour or the other, for both parametric and additive noise.

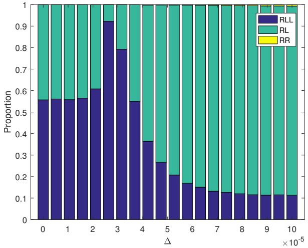
The results have not been entirely as we had expected. The relationship between noise amplitude and the proportion of points going to each of the coexisting attractors is not monotonic for μ in a neighbourhood of μ_m^s.

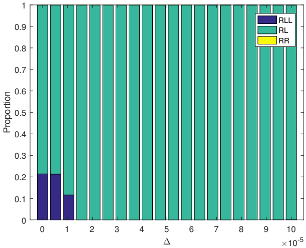


Proportions









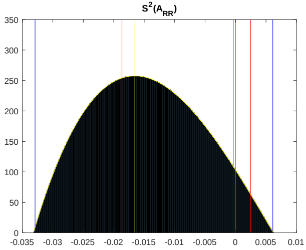
The Transition Mechanism

Perhaps the most interesting phenomenon that we have observed is the potential for repeated intervals of persistent RL dynamics in a noisy system with $\mu < \mu_2^s$.
In the case of both additive and parametric noise, we have observed that the noise-induced transition between RLL and RL behaviour in this case takes the following symbolic form

$$RLLRLL \dots RLL\underline{RLLRLRRL}RL \dots RL. \tag{3}$$

The most significant feature of the transition given in (3) is the repeated R, corresponding to repeated low velocity impacts.

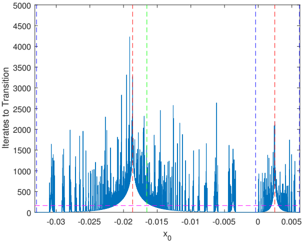
These repeated low velocity impacts allow the dynamics to be pushed into the region of phase space with slow dynamics, in the vicinity of the unstable $(RL)^\infty$ orbit of the deterministic system.

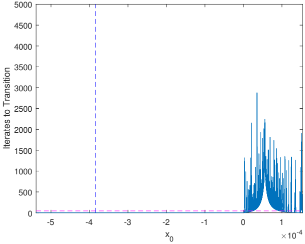


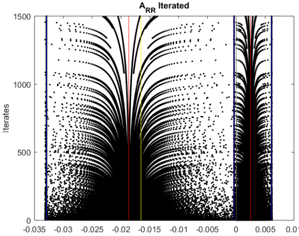
The Transition Mechanism

Let $A_{X_1,X_2,\ldots X_m}$ with $X_i \in \{L, R\}$ for $i \in \{1, 2, \ldots m\}$ denote the set of values x_1 such that the sequence $x_1, x_2, \ldots x_m$ generated under iteration has the symbolic representation $X_1, X_2, \ldots X_m$.

Now since $A_{RR} = \left(0, (\mu/a)^2\right)$ and L_3^2, the second left iterate of the deterministic period-3 orbit, is close to 0 for μ in a neighbourhood of μ_2^s, it is easy to see how a small error due to noise could push the dynamics of a settled RLL orbit into A_{RR}.





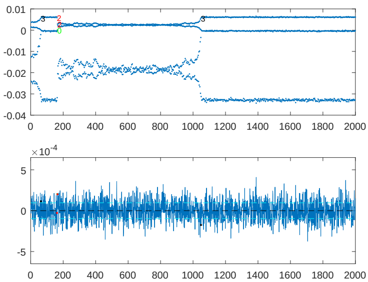


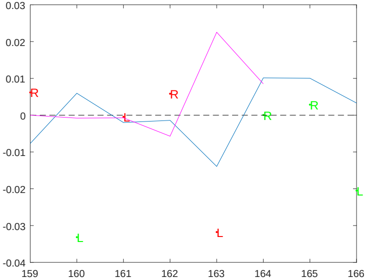
The Transition Mechanism

Let $z_1 = x_1$, $z_{n+1} = S(z_n)$ and $\epsilon_n = x_n - z_n$ for $n \in \{2, 3, \dots\}$.
Given $x_1 \approx R_3$, the right iterate of the stable deterministic period-3 orbit, and the noise terms are such that $|\xi_i| \lll 1$ for all i, it is most likely that the driving force behind such a transition is the error ϵ_4.

$$\epsilon_6 = ab(\sqrt{z_4} - \sqrt{z_4 + \epsilon_4}) + b\xi_4 + \xi_5 \text{ and } \epsilon_4 = b^2\xi_1 + b\xi_2 + \xi_3.$$

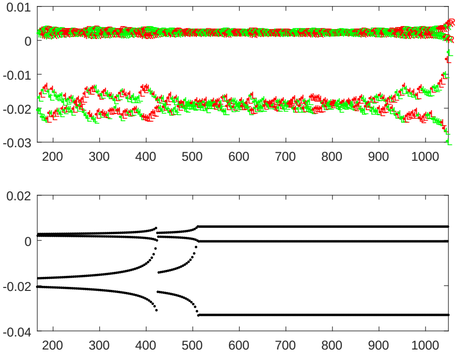
For ϵ_4 to contribute positively to the transition we must have that $\epsilon_4 < 0$.





Holding the unstable stable

Not only can noise cause us to transition from RLL behaviour to RL behaviour in a situation where only the $(RLL)^\infty$ attractor is stable in the deterministic system, but it can also cause orbits to remain in this RL behaviour for longer periods of time than they would in the corresponding deterministic system.

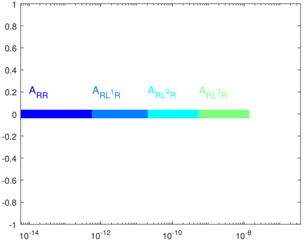


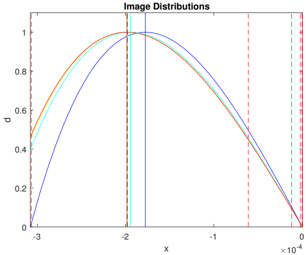
Generalising

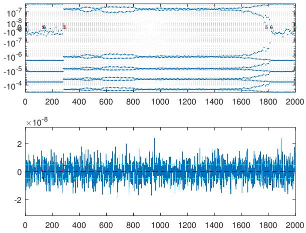
In general we see that these features are repeated as we look at the coexistence of attractors $(RL^{m-1})^\infty$ and $(RL^m)^\infty$ for increasing m. In particular we observe transitions of the following form for μ in a neighbourhood of μ_m^s such that $\mu < \mu_m^s$.

$$RL^m RL^m \dots \underline{RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1}} \dots RL^{m-1} \tag{4}$$

for $k \in \{2, 3, \dots, m\}$.







- W. Chin, E. Ott, H. E. Nusse, and C. Grebogi, *Grazing bifurcations in impact oscillators*, Physical Review E **50** (1994), no. 6, 4427–4444.
- S. J. Linz and M. Lücke, *Effect of additive and multiplicative noise on the first bifurcations of the logistic model*, Physical Review A **33** (1986), no. 4, 2694–2703.
- A.B. Nordmark, *Non-periodic motion caused by grazing incidence in an impact oscillator*, J. Sound Vib. **145** (1991), 279–297.
- D.J.W. Simpson, S.J. Hogan, and R. Kuske, *Stochastic regular grazing bifurcations*, SIADS **12** (2013), 533–559.
- D.J.W. Simpson and R. Kuske, *The influence of localised randomness on regular grazing bifurcations with applications to impact dynamics*, Journal of Vibration and Control **12** (2016), 1077546316642054.
- J.B.W. Webber, *A bi-symmetric log transformation for wide-range data*, Measurement Science and Technology **24** (2013), 027001.