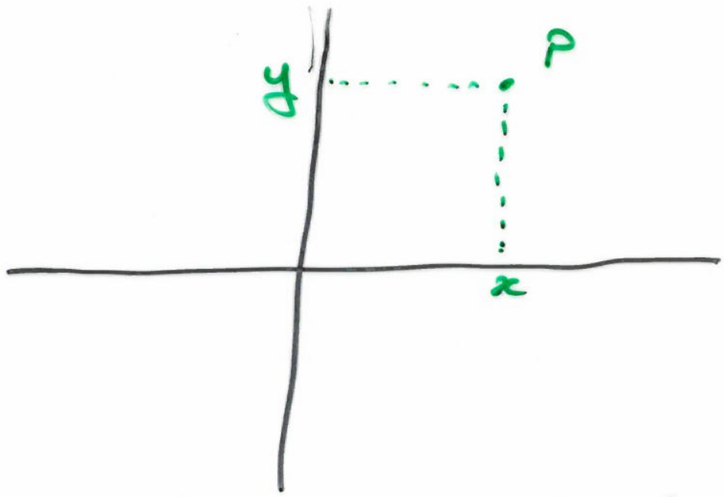
Linear Transformations of the Plane

$\mathbb{R}^2$ denotes the xy-plane.



Any point P in the plane can be represented by a pair of real numbers (x, y).

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

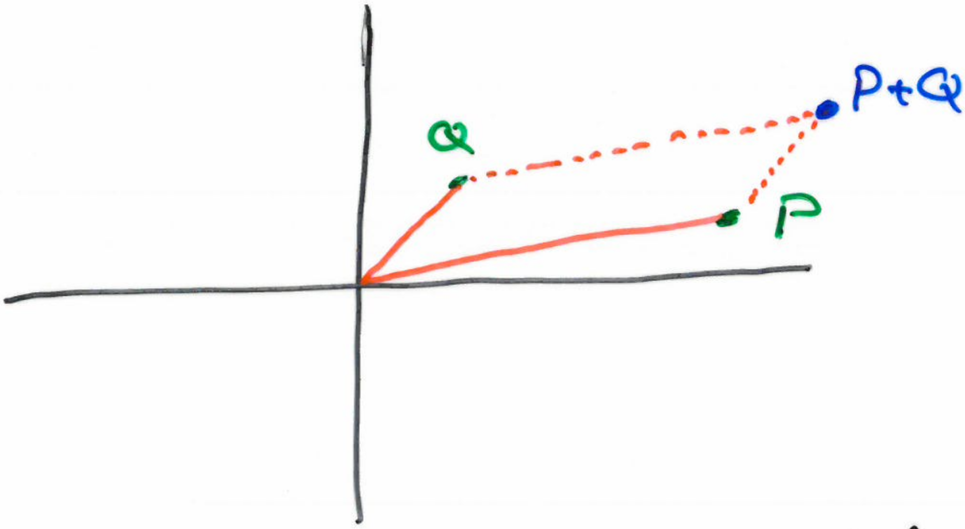
which sends each point $P = (x, y)$ to some point $T(P)$.

we can add two points

$P = (x, y)$ $\qquad$ $Q = (x', y')$

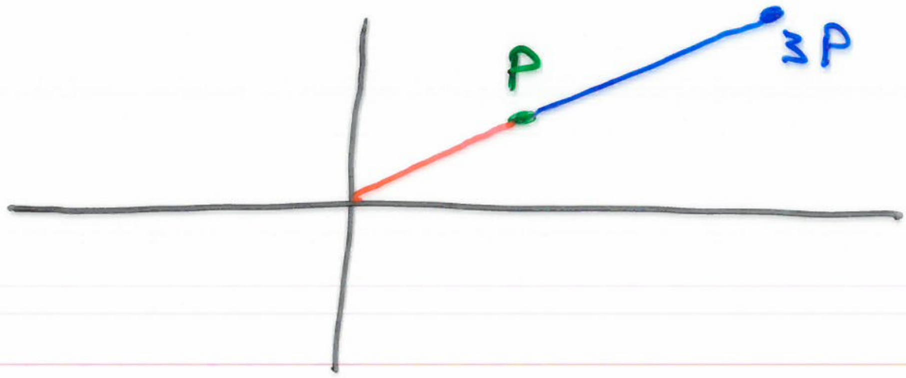
using matrix addition

$$\boxed{P + Q = (x + x', y + y')}.$$



We can multiply a point $P = (x, y)$ by a scalar $\lambda \in \mathbb{R}$ using the formula

$$\boxed{\lambda P = (\lambda x, \lambda y)}.$$



$\lambda = 3$

Definition A transformation

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be linear if:

1) $T(P+Q) = T(P) + T(Q)$

2) $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Example Consider the transformation

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto (3x + 7y,\ 2x + 5y).$$

for instance

$T(1,2) = (17, 12)$

$T(-3,1) = (-2, -1)$

Is T linear?

Consider $P = (x,y)$, $Q = (x', y')$.

$$T(P+Q) = T(x+x', y+y')$$

$$= (3(x+x') + 7(y+y'), 2(x+x') + 5(y+y'))$$

$$= (3x + 7y + 3x' + 7y', 2x + 5y + 2x' + 5y')$$

$$= (3x + 7y, 2x + 5y) + (3x' + 7y', 2x' + 5y')$$

$$= T(P) + T(Q).$$

Also

$$T(\lambda P) = T(\lambda x, \lambda y)$$

$$= (3\lambda x + 7\lambda y, 2\lambda x + 5\lambda y)$$

$$= \lambda(3x + 7y, 2x + 5y)$$

$$= \lambda T(P).$$

Thus T is a linear transformation.

<u>Example</u> Consider

$T : \mathbb{R}^2 \to \mathbb{R}^2, \; (x, y) \mapsto (x^2, y^2)$

e.g. $T(1,2) = (1, 4)$

Is T linear ?

Consider

$$P = (7, 9)$$

$$\lambda = 4$$

$T(\lambda P) = T(28, 36) = (784, 1296)$

$\lambda T(P) = 4.(49, 81) = (196, 324)$

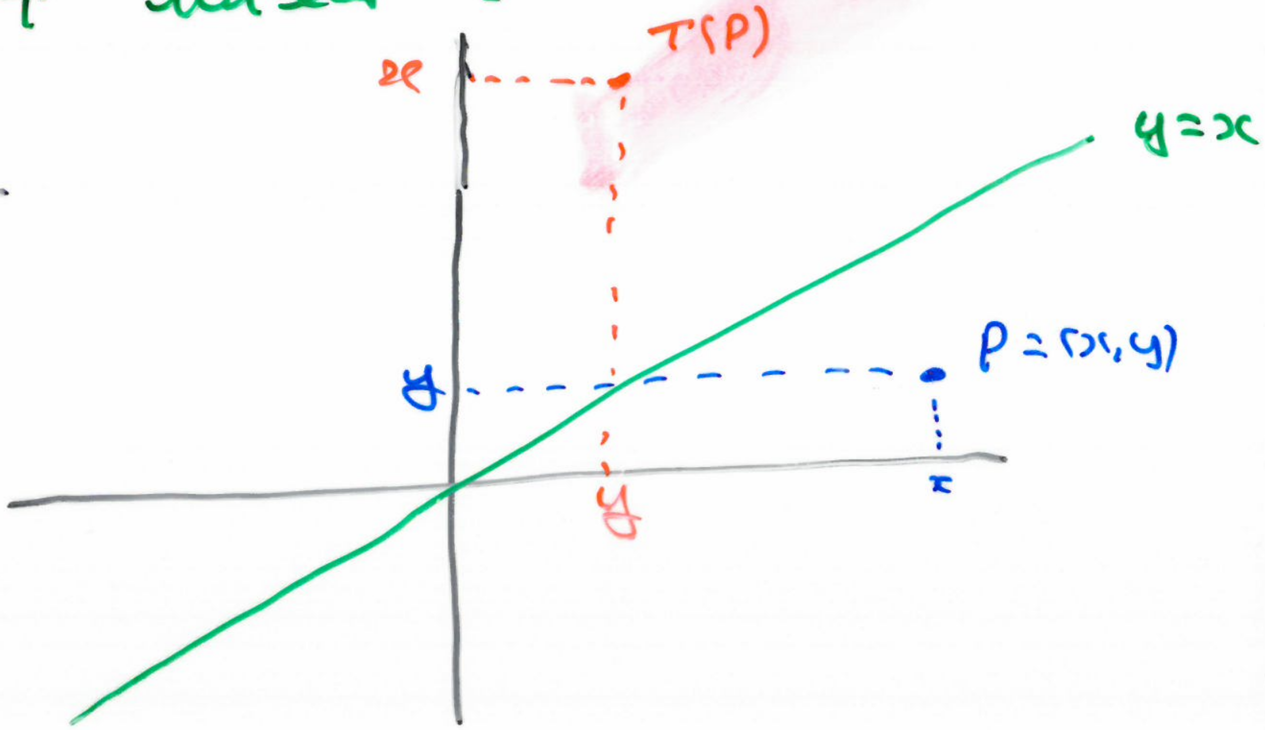
So for this choice of P, λ we have $T(\lambda P) \neq \lambda T(P)$.

Hence T is <u>not</u> linear.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of the plane obtained by reflecting in the line $y = x$.

Is T linear ?

<u>Sol^n</u>



$T(x, y) = (y, x)$.

Is T linear ?

Consider $P = (x, y)$, $Q = (x', y')$

$$T(P+Q) = (x+x', y+y')$$

$$= (y+y', x+x')$$

$$= (y, x) + (y', x')$$

$$= T(P) + T(Q)$$

$$T(\lambda P) = T(\lambda x, \lambda y)$$

$$= (\lambda y, \lambda x)$$

$$= \lambda (y, x)$$

$$= \lambda T(P).$$

Thus, reflection in the line $y = x$ <u>is</u> a linear transformation of the plane.