Yesterday: we checked that

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (3x+7y, 2x+5y)$

is linear, i.e.

- $T(P+Q) = T(P) + T(Q)$
- $T(\lambda P) = \lambda T(P)$ for $\lambda \in \mathbb{R}$.

Note:

$T(x,y) \mapsto (3x+7y, 2x+5y)$

can be represented as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+7y \\ 2x+5y \end{pmatrix}$$

We say that the matrix

$$\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$$

represents T.

Theorem Any linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Proof Let T be any linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

Well $T(1,0) = (a,b)$ say,

and $T(0,1) = (c,d)$ say.

$$T(x,y) = T\big(x(1,0) + y(0,1)\big)$$

by linearity

$$= T(x.(1,0)) + T(y.(0,1))$$

$$= x\,T(1,0) + y\,T(0,1)$$

$$= x.(a,b) + y.(c,d)$$

$$= (xa, xb) + (yc, yd)$$

$$= (xa+yc, xb+ya).$$

Now

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xa+yc \\ xb+yd \end{pmatrix}$$

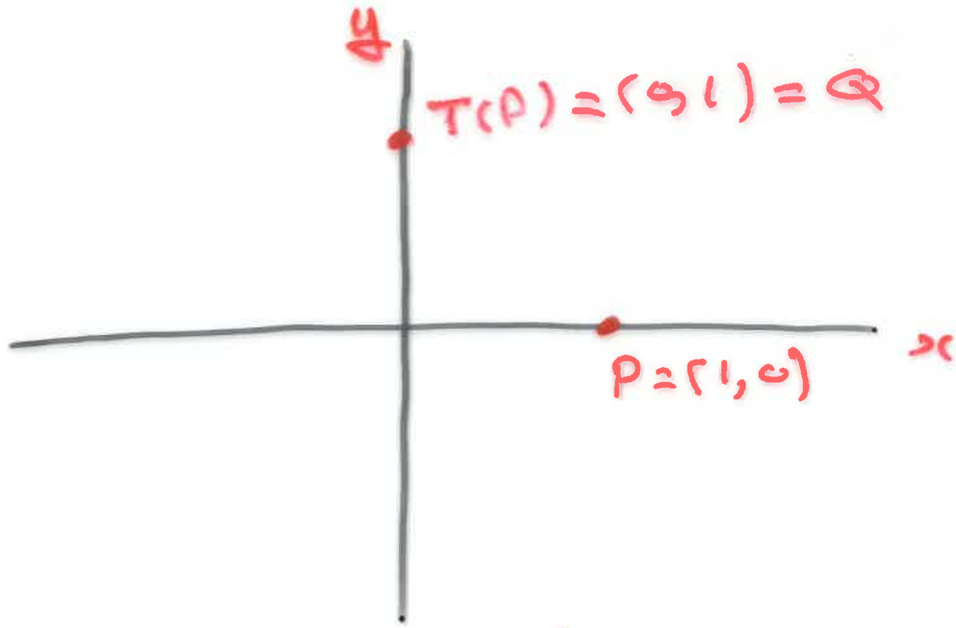
QED

Fact:

- Any reflection in a line through the origin is linear.
- Any rotation of the plane about the origin is linear.
- Any composite of linear transformations is linear.

Example Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ radians about the origin.

Sol^n



$$T(1,0) = (\overset{a}{0}, \overset{b}{1})$$

$$T(0,1) = (\underset{c}{1}, \underset{d}{0})$$

So T is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem Let

$S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$

be linear transformations

represented by matrices

A and B.

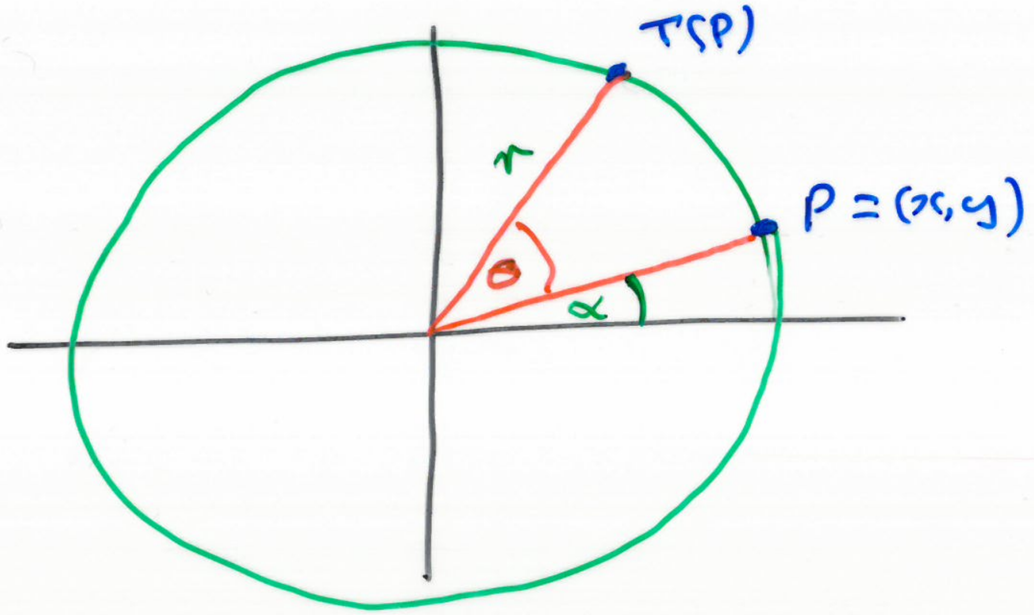
Then the linear transformation

$S \circ T: \mathbb{R}^2 \to \mathbb{R}^2, \; v \mapsto S(T(v))$

is represented by the matrix

AB.

Consider an anticlockwise rotation of the plane about the origin through an angle θ. What matrix represents this transformation?



If $P = (x, y) = (r\cos\alpha,\ r\sin\alpha)$

then

$T(P) = (r\cos(\theta+\alpha),\ r\sin(\theta+\alpha))$

$\quad = r(\cos(\theta+\alpha),\ \sin(\theta+\alpha))$

$$= r(\cos\alpha\cos\theta - \sin\alpha\sin\theta,\ \sin\alpha\cos\theta + \sin\theta\cos\alpha)$$

$$= (x\cos\theta - y\sin\theta,\ y\cos\theta + x\sin\theta)$$

So

$$T(P) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\substack{\text{matrix of anticlock} \\ \text{rotation} \\ \text{about origin} \\ \text{through an} \\ \text{angle } \theta.}} \begin{pmatrix} x \\ y \end{pmatrix}$$